An Infrared Reconstruction Consistency Theorem

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Abstract

We state the strongest common conclusion supported by a reconstruction from microscopic distinguishability data to a local Lorentzian infrared theory. The conclusion is an implication inside a defined class, not an existence theorem for the reconstruction and not a theorem that every quantum system generates general relativity. The antecedent specifies a faithful constant-rank state domain and one monotone information metric; a smooth, local, natural Lorentzian reconstruction modulo diffeomorphisms; a complete physical tangent map; a well-posed diffeomorphism-invariant infrared functional; anomaly and boundary control; one nondegenerate massless metric spin-two sector; and a controlled derivative expansion. Stationarity of the pulled-back functional then forces the full reconstruction defect to vanish. Its leading local zero- and two-derivative part is $G_{\mu\nu} + \Lambda g_{\mu\nu} - 8\pi G_{\mathrm{N}} T_{\mu\nu}$, while local higher-derivative tensors and nonlocal massless-loop form factors remain as specified corrections. We give both an exact tensor equation and a smeared error estimate that does not use an indefinite Lorentzian pointwise norm. The proof is decomposed into reconstruction, variational, Ward, spectrum, and effective-field-theory lemmas, with a dependency graph showing where the entanglement, metric-beta-functional, and response-stiffness routes require additional dictionaries. Removing individual hypotheses produces explicit countermodels, including finite quantum systems without spacetime, restricted metric tangents, scalar-tensor theories, anomalous or nonlocal functionals, and flows with no variational relation. A finite Haskell model checks the dependency algebra and correction bounds; those checks illustrate the logic but do not prove the continuum hypotheses.

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1 Introduction

The claim that spacetime dynamics follow from quantum information can mean several inequivalent things. A family of density operators can carry a monotone Riemannian metric. A localization structure and probe family can be used to reconstruct a continuum metric. An infrared effective action can govern that metric. Relative entropy can provide a linearized constraint in a holographic or causal-diamond dictionary.

A reconstruction scale can admit a beta functional. A quadratic response can identify the coefficient of a physical spin-two kinetic term. Each statement has a well-defined domain; none of them, taken alone, entails the others.

This paper assembles the implications established in the preceding parts of the series into one theorem. Its logical form is

$$\begin{gathered}\text{specified microscopic datum and reconstruction,}\\ \text{local Lorentzian infrared field space modulo gauge,}\\ \text{complete variational tangent and stationary effective action,}\\ \text{Ward-compatible one-metric low-energy expansion}\\ \Downarrow\\ \text{vanishing reconstruction defect,}\\ \text{Einstein form at leading local derivative order,}\\ \text{specified local and nonlocal corrections.}\end{gathered} \tag{1}$$

The antecedent is substantial. In particular, the theorem neither constructs the reconstruction map nor proves that a generic microscopic system has a massless spin-two mode. A gapped spin system, topological phase, quantum mechanical matrix algebra, or continuum theory with a different long-range spectrum can satisfy ordinary quantum-information identities while lying outside the theorem.

The result is a universality statement about form. Within the admitted class, the local pure-metric action through two derivatives is Einstein–Hilbert plus the cosmological term, and its normalized metric equation has the familiar $8\pi G_{\mathrm{N}}$ coupling to the stress tensor. The numerical values of G_{N} and Λ, the matter content, the boundary state, and all higher Wilson coefficients remain model dependent. The cosmological term is leading rather than derivative suppressed. Massless loops generate nonanalytic form factors that cannot be replaced by a finite local operator basis. Additional light fields must be retained rather than hidden in an error term.

Several established results motivate different parts of the antecedent. Petz's classification controls monotone quantum metrics [1, 2]. Consistent massless spin-two coupling and low-energy gravity motivate the spectrum and effective-action assumptions [3, 4, 5]. Diffeomorphism identities follow from covariant variational calculus [6]. Holographic and causal-diamond arguments can supply linearized constraints only with their own entropy and stress dictionaries [15, 16, 17]. Nonlinear sigma models provide a genuine example in which a metric beta function participates in target-space equations [21, 22]; this is a precedent, not a theorem about arbitrary reconstruction scales.

The main contribution is therefore organizational and deductive. We expose the hypotheses in one place, prove the exact implication, quantify the leading error, and show with countermodels why each bridge is needed. This makes the claim refutable in a concrete microscopic proposal: one can test the reconstruction, its tangent rank, its Ward identities, its spectrum, and its scale window independently.

1.1 Results at a glance

The central conclusions are as follows.

(1) Stationarity of the action pulled back to reconstruction space gives the local metric equation only when the reconstruction tangent separates all physical local tensor covectors. Otherwise it gives a projection.

(2) Diffeomorphism invariance makes the defect compatible with the Ward identity after all retained nonmetric equations, source forces, anomalies, boundary fluxes, and gauge directions have been handled. Conservation does not make a nonzero defect vanish.

(3) In the one-metric, local, parity-even, controlled low-energy class with $d > 2$, the local pure-metric action through two derivatives has Einstein–Hilbert form. In $d = 4$ this yields the usual leading tensor equation; lower dimensions and enlarged spectra need separate conclusions.

(4) The exact equation retains $H^{\mathrm{loc}}_{\mu\nu}$ from local higher-derivative operators and $H^{\mathrm{nl}}_{\mu\nu}$ from nonlocal terms. A dimensionless smeared estimate records both contributions.

(5) Entanglement equilibrium, beta fixed points, and response stiffness join the theorem only after route-specific dictionaries are supplied. They are neither alternative names for stationarity nor universal consequences of information-metric positivity.

2 Logical domain of the theorem

Let $\mathfrak{D}$ denote a microscopic information datum and let $\mathfrak{R}_\mu$ denote a reconstruction at scale μ. A statement about every $\mathfrak{D}$ would have the form

$$\forall\mathfrak{D}\ \exists(M,g)\ \mathsf{Einstein}(M,g). \tag{2}$$

No such statement is established here. The theorem instead quantifies over records that already satisfy a predicate Adm:

$$\forall(\mathfrak{D},\mathfrak{R}_\mu,\Gamma_{\mathrm{IR}}), \qquad \mathsf{Adm}(\mathfrak{D},\mathfrak{R}_\mu,\Gamma_{\mathrm{IR}}) \Longrightarrow \mathsf{Consistent}(\mathfrak{R}_\mu,\Gamma_{\mathrm{IR}}). \tag{3}$$

The predicate is unpacked in section 4. Its clauses are empirical or model-theoretic obligations, not definitions chosen to make the conclusion tautological. For example, tangent completeness can be falsified by computing the rank or annihilator of the linearized reconstruction map. Spectrum control can be falsified by finding an additional pole. Locality can be falsified by a long-range response kernel not accounted for among retained massless terms.

Definition 2.1 (Reconstruction consistency)**.** A reconstructed infrared configuration is consistent with a specified effective functional if every admitted field equation and boundary balance law holds on that configuration. Its metric component is consistent when the normalized metric Euler covector vanishes on the physical metric variation space. If the physical tangent is complete, this is equivalent to vanishing of the corresponding tensor distribution.

The word "consistency" does not specify a microscopic relaxation mechanism. It denotes compatibility between a reconstruction output and the stationary infrared theory. A nonzero residual may diagnose that the output is off shell, that a coupling has been mismatched, that an omitted field is important, or that the reconstruction dictionary fails. It does not select which of these possibilities occurs, nor does it define physical time evolution.

Definition 2.2 (Leading-order universality)**.** Leading-order universality means that every record in the admitted class has, up to invertible local field redefinitions and a boundary completion, the same local pure-metric operator basis through two derivatives:

$$\Gamma_g^{(\le 2)} = \frac{1}{16\pi G_{\mathrm{N}}}\int_M \mathrm{d}^d x\sqrt{-g}\,(R-2\Lambda). \tag{4}$$

It does not mean that G_{N}, Λ, the state, or the higher-order coefficients take universal numerical values.

This definition is deliberately weaker than ultraviolet uniqueness and stronger than a dimensional analogy. It says that the same tensor structure governs the leading local metric response within a spectrum and scale class. It leaves open how often microscopic systems enter that class.

3 Microscopic and reconstructed data

3.1 One information metric, not an interchangeable family

At scale μ, take the microscopic datum to be

$$\mathfrak{D}_\mu = \big(\mathsf{P}_\mu, \mathcal{A}_\mu, \Theta_\mu, \rho_\mu, f_\mu, \mathsf{C}_\mu\big). \tag{5}$$

Here P_μ is a region poset, $\mathcal{A}_\mu$ is an isotonic net, $\rho_\mu(\lambda)$ is a smooth faithful state family on a fixed-rank stratum, f_μ selects one symmetric operator-monotone function and its normalization, and C_μ contains the declared coarse-graining channels. When Umegaki relative entropy is used, the mixed-Hessian convention gives the Bogoliubov–Kubo–Mori tensor

$$\mathcal{G}^{\mathrm{BKM}}_{ab}(\lambda) = -\,\partial_a \partial_{b'} D\big(\rho(\lambda)\|\rho(\lambda')\big)\big|_{\lambda'=\lambda}. \tag{6}$$

The minus sign is part of the convention. Bures, symmetric-logarithmic-derivative Fisher, BKM, and general Petz metrics are not treated as identical. Changing f_μ changes the microscopic datum.

The information metric is positive on the appropriate tangent quotient. A positive pullback cannot, by itself, produce a Lorentzian negative direction. The reconstructed causal structure below is therefore independent data, such as a clock field, causal order, modular-time prescription, or another explicit Lorentzianization rule.

3.2 The reconstruction interface

Write the reconstruction output as

$$\mathfrak{R}_\mu(\mathfrak{D}_\mu) = \big[M, g, \Phi; \boldsymbol{c}(\mu)\big], \qquad \mathrm{sig}(g) = (-, +, \ldots, +). \tag{7}$$

Square brackets indicate a quotient by declared redundancies. The interface also contains a localization map from microscopic regions to open sets, probe maps for candidate spacetime directions, a causal completion, and a variation map

$$V_{\mu,\lambda} : T_\lambda \Theta_\mu \longrightarrow \Gamma_{\mathrm{loc}}\big(S^2 T^* M \oplus E_\Phi\big). \tag{8}$$

The probe map and the variation map have different jobs. Probe injectivity supports identifiability of reconstructed directions. The range of V says which infrared field variations microscopic perturbations can actually test.

Let $\mathcal{V}_{g,B}$ be the metric variations that preserve boundary data B and let

$$\mathcal{V}^{\mathrm{gauge}}_{g,B} = \{\mathcal{L}_\xi g : \xi \text{ preserves } B\}. \tag{9}$$

The physical metric tangent is $\mathcal{V}^{\mathrm{phys}}_{g,B} = \mathcal{V}_{g,B}/\mathcal{V}^{\mathrm{gauge}}_{g,B}$. Linearizing the metric output gives

$$D\mathfrak{R}^{(g)}_{\mu,q} : T_q \mathcal{Q} \longrightarrow \mathcal{V}_{g,B}. \tag{10}$$

Its adjoint will turn the metric defect into the first variation on reconstruction space.

4 Hypotheses

We now define the admissibility predicate in equation (3). The labels organize the proof; they do not claim that one hypothesis follows from another.

Assumption 4.1 (Microscopic regularity and metric specification)**.** The state family remains on a smooth faithful or specified constant-rank stratum, the local net and channels are fixed, and one monotone information metric with normalization and boundary convention is selected.

Assumption 4.2 (Regular Lorentzian reconstruction)**.** On an open scale interval, equation (7) returns a smooth d-manifold, a nondegenerate Lorentzian metric, and all other retained fields. The localization, probe, causal, and variation maps have the differentiability needed by the infrared expansion. Topology, dimension, signature, and field content do not jump inside the interval.

Assumption 4.3 (Locality and natural redundancy)**.** The reconstruction response is local up to a declared length ℓ_* and explicitly retained gapless kernels. Microscopic relabelings representing the same physics act on output representatives by pullback and admitted local field redefinitions. In particular, $\mathrm{Diff}(M)$ acts faithfully on representatives, and reconstructed observables factor through its physical quotient.

Assumption 4.4 (Physical tangent completeness)**.** The image of equation (10) is dense in $\mathcal{V}^{\mathrm{phys}}_{g,B}$ in a topology for which pairing with the metric Euler distribution is continuous. Equivalently, an admissible defect covector that annihilates every reconstruction-induced metric variation and every gauge direction must vanish on the full physical metric tangent.

Assumption 4.5 (Well-posed variational infrared theory)**.** There is a renormalized infrared functional $\Gamma_{\mathrm{IR}}[g,\Phi]$ with a well-posed first variation for boundary data B. The retained nonmetric field equations hold, or the combined reconstruction tangent is complete in the full retained field space. External sources are fixed when solving equations and are transformed as spurions when deriving Ward identities.

Assumption 4.6 (Stationarity on reconstruction space)**.** For $\widehat{\Gamma}_\mu = \Gamma_{\mathrm{IR}} \circ \mathfrak{R}_\mu$,

$$D\widehat{\Gamma}_{\mu,q}[\delta q] = 0 \qquad \text{for every admissible } \delta q \in T_q\mathcal{Q}. \tag{11}$$

If stationarity is supplied through an entanglement or scale-flow argument, the corresponding dictionary hypotheses in section 11 must also hold.

Assumption 4.7 (Ward, source, anomaly, and boundary compatibility)**.** The total functional and measure are diffeomorphism invariant on the retained field and source content. All nonmetric equations needed by the Ward identity hold, background sources exert no unbalanced force, any gravitational anomaly is canceled or supplied by inflow, and the boundary conditions remove or account for the relevant flux terms.

Assumption 4.8 (Infrared spectrum and dimension)**.** The geometric sector contains one nondegenerate massless spin-two mode represented by $g_{\mu\nu}$ with a positive physical kinetic coefficient. No additional unsuppressed long-range spin-two, torsion, nonmetricity, scalar, or vector mode mixes at the same order unless it is explicitly retained in Φ and in the conclusion. The dimension, parity sector, topology, and boundary class are fixed; $d > 2$ for the Einstein–Hilbert classification, and $d \geq 4$ when local propagating pure-metric polarizations are asserted.

Assumption 4.9 (Controlled infrared expansion)**.** There are scales $\ell_* \ll L$ such that local analytic operators are ordered by derivatives on the backgrounds considered. Their dimensionless Wilson coefficients are bounded so that no coefficient compensates the powers of ℓ_*/L. Nonanalytic form factors from retained massless fields are kept separately and assigned a causal or Euclidean prescription appropriate to the observable.

Definition 4.10 (Admissible reconstruction record)**.** An admissible record for the core theorem is a tuple

$$\mathfrak{X}_\mu = (\mathfrak{D}_\mu, \mathfrak{R}_\mu, \Gamma_{\mathrm{IR}}; B, \ell_*, L) \tag{12}$$

satisfying theorems 4.1 to 4.9.

The information-metric clause fixes the microscopic meaning of the series, but it does not enter the metric variation algebra. The causal completion, gauge action, functional, and spectrum clauses are separate bridges. This separation prevents the positive BKM tensor in equation (6) from being used as though it already contained Lorentzian dynamics.

5 Effective functional and reconstruction defect

Use signature $(-,+,\ldots,+)$ and vary the inverse metric. Then

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}. \tag{13}$$

Split the functional for bookkeeping as

$$\Gamma_{\mathrm{IR}}[g,\Phi] = \Gamma_{\mathrm{g}}[g,\Phi] + \Gamma_{\mathrm{m}}[g,\Phi] + \Gamma_{\mathrm{nl}}[g,\Phi], \tag{14}$$

while remembering that finite counterterms and field redefinitions can move terms among the first two sectors. Define

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta\Gamma_{\mathrm{m}}}{\delta g^{\mu\nu}}, \tag{15}$$

$$\mathcal{E}_{\mu\nu} := \frac{16\pi G_{\mathrm{N}}}{\sqrt{-g}}\frac{\delta(\Gamma_{\mathrm{g}} + \Gamma_{\mathrm{nl}})}{\delta g^{\mu\nu}}, \tag{16}$$

$$\mathcal{C}_{\mu\nu} := \mathcal{E}_{\mu\nu} - 8\pi G_{\mathrm{N}} T_{\mu\nu} = \frac{16\pi G_{\mathrm{N}}}{\sqrt{-g}}\frac{\delta\Gamma_{\mathrm{IR}}}{\delta g^{\mu\nu}}. \tag{17}$$

Thus, at fixed retained matter fields and up to the prescribed boundary terms,

$$\delta\Gamma_{\mathrm{m}} = -\frac{1}{2}\int_M \mathrm{d}^d x\sqrt{-g}\,T_{\mu\nu}\delta g^{\mu\nu}. \tag{18}$$

The defect covector is

$$\mathbf{C}_g(h) := \frac{1}{16\pi G_{\mathrm{N}}}\int_M \mathrm{d}^d x\sqrt{-g}\,\mathcal{C}_{\mu\nu}h^{\mu\nu}. \tag{19}$$

It is this covector, rather than unnormalized components, that appears in the chain rule and transforms naturally under field redefinitions.

The local gravitational functional can be organized as

$$\begin{aligned}\Gamma_g^{\mathrm{loc}} = {} & \frac{1}{16\pi G_{\mathrm{N}}}\int_M \mathrm{d}^d x\sqrt{-g}\,(R-2\Lambda) \\ & + \int_M \mathrm{d}^d x\sqrt{-g}\,\left[a_1 R^2 + a_2 R_{\mu\nu}R^{\mu\nu} + a_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \cdots\right].\end{aligned} \tag{20}$$

Let

$$H_{\mu\nu}^{\mathrm{loc}} := \frac{16\pi G_{\mathrm{N}}}{\sqrt{-g}}\frac{\delta\Gamma_g^{(\geq 4)}}{\delta g^{\mu\nu}}, \tag{21}$$

$$H_{\mu\nu}^{\mathrm{nl}} := \frac{16\pi G_{\mathrm{N}}}{\sqrt{-g}}\frac{\delta\Gamma_{\mathrm{nl}}}{\delta g^{\mu\nu}}. \tag{22}$$

The full defect then has the expansion

$$\mathcal{C}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} + H_{\mu\nu}^{\mathrm{loc}} + H_{\mu\nu}^{\mathrm{nl}} - 8\pi G_{\mathrm{N}} T_{\mu\nu}. \tag{23}$$

When additional light fields are retained in the gravitational sector, their metric Euler contribution must be added explicitly or moved into a declared total stress tensor. It is not automatically higher order.

The local/nonlocal split is scale and scheme dependent in its analytic pieces, but nonanalytic momentum dependence cannot be removed by a finite local counterterm. A representative term is

$$\Gamma_{\mathrm{nl}} \supset \int_M \mathrm{d}^4 x\sqrt{-g}\,R\log\left(\frac{-\Box - i0}{\mu^2}\right)R. \tag{24}$$

The logarithm denotes an integral kernel. Its variation acts on the measure, the curvatures, and the operator itself.

6 Dependency graph and proof decomposition

The proof is acyclic. Information geometry fixes a regular microscopic domain; the reconstruction interface supplies fields and their tangent map; the effective theory supplies a covector; stationarity and completeness make that covector vanish; and the infrared operator classification identifies its leading form. Ward identities control compatibility but do not replace stationarity.

The decomposition used below is:

L1. *Variational pullback*: the derivative of $\Gamma_{\mathrm{IR}} \circ \mathfrak{R}_\mu$ is the adjoint reconstruction tangent applied to $\mathbf{C}_g$.

L2. *Physical separation*: tangent completeness turns vanishing of the pulled-back covector into vanishing on the full physical metric tangent.

L3. *Ward descent*: anomaly-free diffeomorphism invariance makes the defect annihilate gauge directions after nonmetric equations and flux terms are controlled.

L4. *Local operator classification*: locality, dimension, and the single-metric spectrum select equation (4) through two derivatives.

L5. *Correction estimate*: scale separation and coefficient control bound the local higher-derivative remainder; massless nonlocal terms are measured separately.

Entanglement, beta-functional, and stiffness results form side branches, not additional arrows inside figure 1. They can identify a premise or coefficient only under the route-specific hypotheses of section 11.

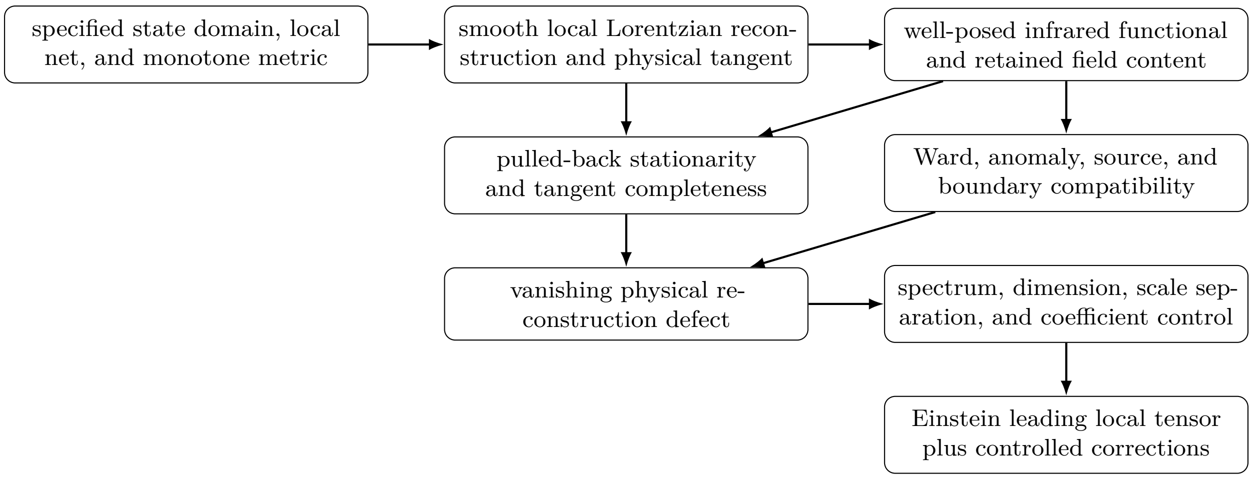

Figure 1: Core dependency graph. The first node does not imply the second; each arrow uses the hypotheses written at its target.

7 Variational and Ward lemmas

Lemma 7.1 (Pullback of the metric Euler covector)**.** *Under theorems 4.2 and 4.5, after the retained nonmetric equations hold, the first variation on reconstruction space is*

$$D\widehat{\Gamma}_{\mu,q}[\delta q] = \mathbf{C}_g\big(D\mathfrak{R}^{(g)}_{\mu,q}[\delta q]\big). \tag{25}$$

Equivalently,

$$D\widehat{\Gamma}_{\mu,q} = \big(D\mathfrak{R}^{(g)}_{\mu,q}\big)^*\mathbf{C}_g. \tag{26}$$

Proof. Apply the functional chain rule to $\Gamma_{\mathrm{IR}}[g(q), \Phi(q)]$. The metric term is

$$\frac{1}{16\pi G_{\mathrm{N}}}\int_M \mathrm{d}^d x\sqrt{-g}\,\mathcal{C}_{\mu\nu}\delta_q g^{\mu\nu} = \mathbf{C}_g(\delta_q g). \tag{27}$$

Every retained nonmetric term is its Euler covector paired with the corresponding component of $D\mathfrak{R}_\mu[\delta q]$. These terms vanish by the nonmetric equations in theorem 4.5; alternatively they are retained in a block covector if the full tangent is used. Fixed external sources do not contribute to the equation variation. This gives equation (25), and the definition of the adjoint gives equation (26). □

Lemma 7.2 (Separation by the reconstruction tangent)**.** *Assume theorem 4.4. If*

$$\big(D\mathfrak{R}^{(g)}_{\mu,q}\big)^*\mathbf{C}_g = 0 \tag{28}$$

and $\mathbf{C}_g$ annihilates gauge directions, then $\mathbf{C}_g = 0$ on $\mathcal{V}^{\mathrm{phys}}_{g,B}$. If $\mathcal{C}_{\mu\nu}$ is smooth, it vanishes pointwise; if it is distributional, it vanishes as a tensor distribution.

Proof. Equation (28) says that $\mathbf{C}_g$ annihilates $\mathrm{Ran}(D\mathfrak{R}^{(g)}_{\mu,q})$. By hypothesis it also annihilates $\mathcal{V}^{\mathrm{gauge}}_{g,B}$. Tangent completeness makes the image dense in the physical variation space, and continuity of the pairing extends the zero to its closure. The fundamental lemma of the calculus of variations for symmetric tensor distributions then gives the stated conclusion. □

The density condition can be weakened if the desired conclusion is itself a projection. It cannot be omitted while retaining the full local tensor equation.

Proposition 7.3 (Projected stationarity)**.** *Without theorem 4.4, reconstruction stationarity implies exactly*

$$\big(D\mathfrak{R}^{(g)}_{\mu,q}\big)^*\mathbf{C}_g = 0, \tag{29}$$

or equivalently $\mathbf{C}_g \in \mathrm{Ann}\,\mathrm{Ran}(D\mathfrak{R}^{(g)}_{\mu,q})$. No full tensor equation follows unless the annihilator is trivial on the physical quotient.

Proof. This is theorem 7.1 with $D\widehat{\Gamma}_{\mu,q} = 0$. A linear map with non-dense range can have a nontrivial annihilator, so the converse implication used in theorem 7.2 is unavailable. ☐

Lemma 7.4 (Ward descent of the defect)**.** *Under theorems 4.5 and 4.7, the full defect obeys*

$$\nabla^\mu \mathcal{C}_{\mu\nu} = 0 \tag{30}$$

on the retained nonmetric equations and admitted boundary domain. Hence $\mathbf{C}_g(\mathcal{L}_\xi g^{-1}) = 0$ *for compactly supported or boundary-preserving* ξ, *and* $\mathbf{C}_g$ *descends to the physical metric quotient.*

Proof. Infinitesimal diffeomorphism invariance of the total functional gives an off-shell identity of the schematic form

$$\nabla^\mu \mathcal{C}_{\mu\nu} = -\mathcal{W}_\nu(\mathcal{E}_\Phi; \Phi) - 8\pi G_{\mathrm{N}} \mathcal{W}^J_\nu + \mathcal{A}_\nu + \mathcal{B}_\nu. \tag{31}$$

Here $\mathcal{W}_\nu(\mathcal{E}_\Phi; \Phi)$ is the formal-adjoint contribution of retained field equations, $\mathcal{W}^J_\nu$ is the force from background sources, $\mathcal{A}_\nu$ is an anomaly, and $\mathcal{B}_\nu$ represents the boundary balance term. Every term on the right vanishes or is canceled under theorem 4.7, giving equation (30). For an inverse-metric gauge variation $\mathcal{L}_\xi g^{\mu\nu} = -2\nabla^{(\mu}\xi^{\nu)}$, integration by parts yields

$$\mathbf{C}_g(\mathcal{L}_\xi g^{-1}) = \frac{1}{8\pi G_{\mathrm{N}}} \int_M \mathrm{d}^d x \sqrt{-g}\, \xi^\nu \nabla^\mu \mathcal{C}_{\mu\nu}, \tag{32}$$

up to the boundary term already controlled. The result vanishes. ☐

Proposition 7.5 (Ward compatibility is not stationarity)**.** *The implication*

$$\nabla^\mu \mathcal{C}_{\mu\nu} = 0 \quad \Longrightarrow \quad \mathcal{C}_{\mu\nu} = 0 \tag{33}$$

is false even for smooth natural tensors.

Proof. For any nonzero constant λ, the tensor $\mathcal{C}_{\mu\nu} = \lambda g_{\mu\nu}$ is nonzero, while metric compatibility gives $\nabla^\mu \mathcal{C}_{\mu\nu} = 0$. It can represent a mismatch in the cosmological coupling. More generally, Euler tensors of diffeomorphism-invariant curvature functionals are divergence compatible but need not vanish off shell. ☐

8 The local infrared classification

Lemma 8.1 (Local metric action through two derivatives)**.** *Assume theorems 4.2, 4.3, 4.8 and 4.9, with* $d > 2$ *and a parity-even analytic local pure-metric sector. Modulo a boundary term and invertible perturbative local field redefinitions, its zero- and two-derivative action is equation (4).*

Proof. Naturality and the faithful diffeomorphism redundancy restrict local terms to integrals of scalar densities. Without extra background tensors, the only zero-derivative pure-metric density is $\sqrt{-g}$. At two derivatives, the only parity-even scalar linear in curvature is R, up to a divergence. The spectrum hypothesis requires a nondegenerate coefficient for the long-range spin-two kinetic term. Its normalization defines G_{N}. The cosmological coefficient defines Λ. The derivative expansion places curvature-squared and higher operators in the local remainder; the coefficient bound prevents them from returning at leading order. ☐

Lovelock's theorem gives a related tensor classification under its own second-order and naturality hypotheses [7]. It must be read dimension by dimension. The Gauss–Bonnet density is topological in four dimensions but contributes a nontrivial second-order tensor in dimensions above four. Its curvature order still makes it a higher-order term in a weak-curvature EFT unless its coefficient is enhanced.

Lemma 8.2 (Local derivative estimate)**.** *Let the background satisfy* $R_{\mu\nu\rho\sigma} = \mathcal{O}(L^{-2})$ *and* $\nabla = \mathcal{O}(L^{-1})$. *For a local operator with* $2n$ *derivatives,* $n \geq 2$, *let* B_{2n} *bound its dimensionless coefficient and field-amplitude factor in a fixed physical smearing norm. Then*

$$\frac{\|H^{(2n)}\|_*}{K_L} \leq B_{2n} \left(\frac{\ell_*}{L}\right)^{2n-2}, \tag{34}$$

where K_L *is a nonzero reference scale for the two-derivative tensor on the same background. More precisely, choose a nonzero two-derivative reference covector* $X^{(2)}_L$ *in the same dual physical smearing space and set* $K_L := \|X^{(2)}_L\|_* > 0$*; numerator and denominator then have identical dimensions.*

Proof. Relative to the Einstein–Hilbert tensor, every additional derivative pair contributes L^{-2}, while the coefficient supplies the matching power of ℓ_*^2. The bounded dimensionless coefficient and field-amplitude factor give equation (34). The estimate applies to the analytic local expansion and not to the nonanalytic form factors in equation (24). □

The reference K_L is not required to be the pointwise norm of the Einstein tensor. On a background where that tensor vanishes, it may instead be the natural two-derivative response scale L^{-2} times the normalization of the chosen test space. This avoids dividing by an accidentally vanishing on-shell tensor.

9 Infrared reconstruction consistency theorem

Theorem 9.1 (Infrared reconstruction consistency)**.** *Let $\mathfrak{X}_\mu$ be an admissible reconstruction record in the sense of theorem 4.10. Then:*

(i) the full physical defect covector vanishes;

(ii) the smooth reconstructed metric satisfies

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + H^{\mathrm{loc}}_{\mu\nu} + H^{\mathrm{nl}}_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}; \tag{35}$$

(iii) at leading local zero- and two-derivative order, the equation has the universal form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}, \tag{36}$$

with local higher-derivative and nonlocal corrections retained according to equation (35) and section 10.

The theorem asserts neither that every microscopic datum admits such a record nor that the information metric determines G_{N} or Λ.

Proof. Stationarity in theorem 4.6 and the pullback identity theorem 7.1 give $(D\mathfrak{R}^{(g)}_{\mu,q})^* \mathbf{C}_g = 0$. The Ward lemma theorem 7.4 shows that the covector annihilates gauge directions and is well defined on the physical quotient. Tangent completeness and theorem 7.2 therefore imply $\mathbf{C}_g = 0$. Smoothness upgrades this to $\mathcal{C}_{\mu\nu} = 0$ pointwise. Substitution of equation (23) gives equation (35).

The local classification theorem 8.1 identifies the Einstein–Hilbert and cosmological pieces. The matter sign and $8\pi G_{\mathrm{N}}$ factor follow from equations (15) and (18). Removing the local higher-derivative terms at leading order gives equation (36); this removal is an EFT truncation justified by theorem 8.2, not an exact deletion from the full equation. Nonlocal massless terms are not part of that analytic truncation and remain in equation (35) whenever relevant. □

Remark 9.2 (Strength and boundary of the conclusion). The strongest universal content is the leading local tensor structure inside the admitted class. Existence of a smooth Lorentzian reconstruction, tangent completeness, the massless spectrum, stationarity, anomaly cancellation, and coefficient control are hypotheses. A microscopic theory outside this class is not a counterexample to the implication; it is a counterexample to any attempt to erase the antecedent.

Corollary 9.3 (Projected reconstruction equation)**.** *If every hypothesis of theorem 9.1 except theorem 4.4 holds, then the justified conclusion is*

$$\left(D\mathfrak{R}^{(g)}_{\mu,q}\right)^* \left[G + \Lambda g + H^{\mathrm{loc}} + H^{\mathrm{nl}} - 8\pi G_{\mathrm{N}} T\right] = 0. \tag{37}$$

It is generally weaker than the full tensor equation.

Proof. Use theorems 7.1 and 7.3. The EFT expansion identifies the tensor inside the brackets but does not enlarge the range of the reconstruction tangent. □

Corollary 9.4 (Exact Einstein sector)**.** *If, in addition, the admitted functional has no local operators above two derivatives in the working domain and no relevant nonlocal metric term, then equation (36) is exact for the retained fields. This is a restriction on the effective functional, not a generic consequence of taking L large.*

10 Quantitative leading-order statement

A Lorentzian pointwise contraction of a symmetric tensor need not be positive. To say that the defect is small, choose a positive norm $\|\cdot\|_{\mathcal{V}}$ on a gauge-fixed or quotient space of compactly supported physical test tensors. For a tensor distribution X, define the dual smearing norm

$$\|X\|_* := \sup_{\substack{h \in \mathcal{V}^{\mathrm{phys}}_{g,B} \\ \|h\|_{\mathcal{V}}=1}} \left| \int_M \mathrm{d}^d x \sqrt{-g}\, X_{\mu\nu} h^{\mu\nu} \right|. \tag{38}$$

The choice may be a Sobolev norm on a compact region, a Euclideanized gauge-fixed norm, or another operational smearing norm. It is part of the error statement.

Let

$$\varepsilon_{\mathrm{loc}}(L) := \sum_{n\geq 2} B_{2n} \left(\frac{\ell_*}{L}\right)^{2n-2}, \tag{39}$$

$$\varepsilon_{\mathrm{nl}}(L) := \frac{\|H^{\mathrm{nl}}\|_*}{K_L}. \tag{40}$$

The first quantity is an analytic derivative estimate; the second measures the actual nonlocal contribution in the chosen state and prescription. No power of ℓ_*/L is assigned to $\varepsilon_{\mathrm{nl}}$ without a separate calculation.

Theorem 10.1 (Leading-order universality with corrections)**.** *Under the hypotheses of theorem 9.1, assume the coefficient bounds in equation (39) are summable at the stated truncation order. Then the leading Einstein residual*

$$\mathcal{E}^{(2)}_{\mu\nu} := G_{\mu\nu} + \Lambda g_{\mu\nu} - 8\pi G_{\mathrm{N}} T_{\mu\nu} \tag{41}$$

satisfies

$$\|\mathcal{E}^{(2)}\|_* \leq K_L\big(\varepsilon_{\mathrm{loc}}(L) + \varepsilon_{\mathrm{nl}}(L)\big). \tag{42}$$

Consequently, if both correction parameters tend to zero along a controlled infrared family, the leading residual tends to zero in the dual physical smearing norm.

Proof. The exact equation gives $\mathcal{E}^{(2)} = -H^{\mathrm{loc}} - H^{\mathrm{nl}}$. Apply the triangle inequality. Summing equation (34) over retained local orders gives $\|H^{\mathrm{loc}}\|_*/K_L \leq \varepsilon_{\mathrm{loc}}$. The definition equation (40) gives the other term. □

This estimate is the strongest defensible leading-order universality statement without computing the microscopic coefficients. It says how the universal operator form is approached. It does not predict a small cosmological constant, because $\Lambda g_{\mu\nu}$ is included in $\mathcal{E}^{(2)}$. It does not call a light scalar a correction, because theorem 4.8 requires such a scalar to appear among retained fields.

10.1 Correction and obstruction ledger

Not every failure can be folded into the right side of equation (42). The following distinctions are important.

Item	Mathematical representation	Status in the theorem
Local higher derivatives	H^{loc} from R^2, $R^2_{\mu\nu}$, Lovelock, and higher operators	Analytic EFT correction controlled by $\varepsilon_{\mathrm{loc}}$
Massless propagation	H^{nl} from logarithms and other nonanalytic form factors	Nonlocal correction measured by $\varepsilon_{\mathrm{nl}}$
Additional light field	Coupled Euler block or declared total stress tensor	Leading retained physics, not automatically a small correction
Incomplete tangent	Nontrivial $\mathrm{Ann}\,\mathrm{Ran}(D\mathfrak{R}^{(g)})$	Weakens the conclusion to a projection; it is not a tensor correction
Diffeomorphism anomaly	$\mathcal{A}_\nu$ in equation (31)	Obstructs gauge compatibility unless canceled or matched by inflow

Item	Mathematical representation	Status in the theorem
Boundary flux	Surface term or balance current	Requires an enlarged boundary equation; not a bulk local Wilson coefficient
Scheme or field change	Adjoint Jacobian acting on Euler covectors	Changes representatives while preserving a regular common zero locus

If an application assigns numerical tolerances to projection, anomaly, or boundary residuals, it may write an enlarged diagnostic budget. Such a budget measures departure from the theorem's antecedent; it is not the same as the controlled EFT remainder proved in theorem 10.1.

11 Three auxiliary routes and their dictionaries

The series studies three ways to supply or interpret pieces of the core theorem. Their conclusions can agree on a shared record, but their premises remain distinct.

11.1 Relative entropy and entanglement equilibrium

For a faithful reference state σ,

$$D(\rho\|\sigma) = \Delta\langle K_\sigma\rangle - \Delta S, \qquad K_\sigma = -\log\sigma. \tag{43}$$

The first variation at $\rho = \sigma$ gives $\delta S = \delta\langle K_\sigma\rangle$; the second variation gives the BKM quadratic form in its regular domain. To obtain a gravitational constraint, one additionally needs a reference geometry, geometric entropy functional, stress dictionary, state-to-field map, and a complete family of regions or frames.

In holographic CFTs, the Ryu–Takayanagi or generalized-entropy dictionary and the Iyer–Wald identity convert first laws for all appropriate balls into a bulk equation linearized about the reference background [14, 15, 16]. In the local causal-diamond route, fixed-volume entanglement equilibrium and a local modular response yield an Einstein constraint under the small-ball assumptions [17]. Higher-curvature versions use the corresponding entropy and generalized volume [18]. Because these are first-order state variations, a term proportional to the metric and independent of the varied state is not fixed locally. The Bianchi and matter Ward identities make its coefficient spacetime constant on the connected domain, where it is interpreted as a cosmological term. This does not predict that the constant vanishes or is small.

Proposition 11.1 (Entanglement route into the theorem)**.** *Suppose an entanglement-gravity dictionary on a complete perturbation family establishes the linearized condition*

$$\delta\mathbf{C}_g(h) = 0 \tag{44}$$

for all physical test variations about a reference solution. Then the linearization of theorem 9.1 holds on that shared perturbative domain. This does not supply nonlinear stationarity or construct the reconstruction outside the dictionary.

Proof. Completeness of the region or frame family separates the linearized defect in the same way that theorem 4.4 separates the nonlinear first variation. The effective-action classification then identifies its leading tensor. The argument is confined to the perturbative order at which the dictionary and entropy identity were established. □

11.2 Metric beta functionals

A scale-indexed metric is not yet a metric beta functional. One must specify a comparison connection across scales, quotient redundant field directions, construct a local reconstruction RG, and relate its physical beta vector to the local Wilsonian defect covector. In a fixed scheme, the needed relation is

$$[\widehat{\beta}_g] = -\mathcal{M}_g[\mathbf{C}_g^{\mathrm{loc}}], \tag{45}$$

with nonmetric mixing controlled and $\mathcal{M}_g$ injective on the physical defect space. The local-RG framework itself has independent Weyl consistency and source-mixing structure [23]; those consistency conditions do not by themselves prove the variational relation displayed here.

Proposition 11.2 (Beta route into the theorem)**.** *Under the comparison, local RG, Ward, mixing, common-scheme, and injectivity hypotheses just stated,*

$$[\widehat{\beta}_g] = 0 \quad \Longleftrightarrow \quad [\mathbf{C}_g^{\mathrm{loc}}] = 0. \tag{46}$$

With tangent completeness, a physical beta fixed point therefore satisfies the local part of equation (35). The nonlocal term follows only if a separately specified injective nonlocal relation replaces equation (45).

Proof. One implication follows directly from equation (45). For the converse, a zero beta lies in the kernel of $\mathcal{M}_g$; injectivity makes the physical local defect vanish. Tangent completeness converts the covector statement to the tensor statement. □

The proposition does not assert a universal identity between a coarse-graining beta and the Einstein tensor. Raw coupling running, Callan–Symanzik scale independence, a local Weyl equation, and an actual reconstructed metric flow are different operations. The sigma-model analogy is precise only because the target metric is a world-sheet coupling and the coupled background equations have been derived in a specified scheme.

11.3 Quadratic response stiffness

The response route expands the action about a solution and projects away gauge zero modes. If h^{phys} is a physical metric perturbation, the two-derivative quadratic kernel has the schematic low-momentum form

$$\delta^2\Gamma_{\mathrm{EH}} \sim \frac{1}{G_{\mathrm{N}}}\langle h^{\mathrm{phys}}, k^2 h^{\mathrm{phys}}\rangle, \tag{47}$$

up to convention-dependent numerical factors, background curvature, and the chosen gauge-fixed pairing. This makes $1/G_{\mathrm{N}}$ the coefficient of the physical metric response.

The conventions fixed in Part VII make the flat-space factor explicit. Take $g = \eta + h$ in $d \geq 4$, remove proper diffeomorphisms, choose normalized transverse-traceless polarizations $e^s_{ij}e^{s'\,ij} = \delta^{ss'}$, and write the Euclidean quadratic action as $I^{(2)} = \frac{1}{2}\sum_s \int_k q_s^* \mathcal{K}_s q_s$. At nonzero Euclidean momentum, with $k^2 := k_\tau^2 + \delta^{ij}k_ik_j > 0$, the Einstein–Hilbert term gives

$$\mathcal{K}_s(k) = \frac{k^2}{32\pi G_{\mathrm{N}}}, \qquad q_s(k) = \frac{16\pi G_{\mathrm{N}}}{k^2}\, T_s^{\mathrm{TT}}(k), \tag{48}$$

where the second formula uses the stress coupling convention of equation (18); Fourier and Wick-rotation choices can change its sign but not its magnitude. On an Einstein background, the corresponding physical operator has the form

$$\mathcal{K}_{\mathrm{TT}} = \frac{1}{32\pi G_{\mathrm{N}}}\, \mathcal{P}_{\mathrm{TT}}\left[-\bar{\nabla}^2 + \mathcal{U}(\bar{R})\right]\mathcal{P}_{\mathrm{TT}}, \tag{49}$$

so its principal symbol retains the same $1/(32\pi G_{\mathrm{N}})$ normalization while curvature, boundary conditions, and global zero or negative modes control the lower-order spectrum. Neither formula is a positivity statement about the unreduced Euclidean metric Hessian. Higher-derivative terms, nonlocal form factors, or mixed light fields can dominate the observable response outside the Einstein-dominated window.

Calling this coefficient microscopic information stiffness requires further dictionary data. In the holographic vacuum-ball construction these include a semiclassical code subspace, a differentiable state-to-field map into on-shell bulk perturbations, the normalized bulk action, the relevant gauge and boundary conditions, and a controlled semiclassical order. Only on the resulting reconstructible tangent space does the BKM relative-entropy Hessian equal gravitational canonical energy and inherit its $1/G_{\mathrm{N}}$ normalization [19, 20].

Proposition 11.3 (Stiffness interpretation)**.** *For a normalized operational metric, a background solution, well-posed boundary data, a physical quotient by proper diffeomorphisms, and an Einstein-dominated momentum window, the normalization $1/G_{\mathrm{N}}$ in equation (4) is the two-derivative physical response stiffness, with the flat-space convention given by equation (48). If, in addition, the complete holographic dictionary just stated identifies the BKM Hessian with canonical energy, the same coefficient is an information-geometric stiffness on the restricted reconstructible tangent space. Without that additional Hessian dictionary, no generic microscopic Fisher identification follows; in either case theorem 9.1 leaves the numerical value of G_{N} undetermined.*

The precise kernel, zero-mode treatment, and positivity domain determine whether this hypothesis holds in a concrete model. Its role in the main theorem is interpretive: the core proof uses the nonzero kinetic coefficient but not a microscopic formula for it.

11.4 Comparison of the routes

Route	What it can supply	Additional data that cannot be omitted
Effective action	Full stationary defect equation	Local Lorentzian field, complete tangent, well-posed action, field equations
Entanglement	Linearized constraint in a reference domain	Entropy, modular, stress, geometry, and completeness dictionaries
Metric beta	Local defect zero at a physical fixed point	Comparison connection, local RG, common scheme, variational map, injectivity
Stiffness	Meaning of $1/G_{\mathrm{N}}$ as a response coefficient	Gauge quotient, physical Hessian, positive domain, microscopic/canonical-energy dictionary

Table 2: The common defect is a meeting point only after the stated route-specific dictionaries are supplied.

12 Field redefinitions, schemes, and invariant content

Let $\Psi^A = (g, \Phi)$ denote all retained fields and let $\Psi'^A = F^A(\Psi)$ be an invertible local field redefinition with Jacobian $J^A{}_B$. The Euler derivative is a covector. Suppressing spacetime kernels,

$$\frac{\delta\Gamma}{\delta\Psi'^A} = (J^{-1})^B{}_A \frac{\delta\Gamma}{\delta\Psi^B}. \tag{50}$$

For derivative-dependent redefinitions the transpose is a formal adjoint after integration by parts. A nonsingular change therefore preserves the common zero locus of the full Euler block, although individual components and their derivative order can change. The equivalence theorem gives the corresponding on-shell statement for suitable local changes of quantum fields [8].

At fixed EFT order, a perturbative metric redefinition such as

$$g_{\mu\nu} \mapsto g_{\mu\nu} + u\ell_*^2 R_{\mu\nu} + v\ell_*^2 R g_{\mu\nu} \tag{51}$$

moves coefficients among curvature-squared and matter-curvature operators. The statement that the expansion is controlled is invariant under a regular order-by-order redefinition, while a named coefficient in a redundant basis need not be.

Similarly, moving a finite local counterterm I_{ct} between sectors,

$$\Gamma'_{\mathrm{g}} = \Gamma_{\mathrm{g}} + I_{\mathrm{ct}}, \qquad \Gamma'_{\mathrm{m}} = \Gamma_{\mathrm{m}} - I_{\mathrm{ct}}, \tag{52}$$

changes $\mathcal{E}_{\mu\nu}$ and $T_{\mu\nu}$ separately but leaves the total defect equation (17) unchanged. Therefore the theorem concerns a fixed total renormalized functional and its physical Euler zero locus, not an absolute off-shell split between “geometry” and “matter.”

Proposition 12.1 (Covariance of the consistency theorem)**.** *Under an invertible, scale-independent local redefinition that maps the admitted boundary domain and physical quotient regularly, the vanishing full Euler block and the conclusion of theorem 9.1 are preserved. The representative form of H^{loc} and the numerical error constants may change.*

Proof. Equation (50) and invertibility preserve simultaneous Euler zeros. The transformed reconstruction tangent is the Jacobian pushforward of the original tangent, so density on the corresponding quotient is preserved. Natural field redefinitions carry the Ward identity and boundary domain to their transformed forms. The operator basis and smearing norm can change, which accounts for the final qualification. □

Explicitly scale-dependent field coordinates acquire an inhomogeneous term in their beta transformation. Consequently, a raw zero of components need not remain a raw zero under such a change. The beta route in theorem 11.2 avoids this problem by using a comparison connection and a physical quotient.

13 Failure modes and countermodels

The implication in theorem 9.1 is most informative when its antecedent can be challenged. The following examples show that ordinary information identities or subsets of the infrared hypotheses cannot replace the missing bridge.

13.1 A faithful quantum system without reconstructed spacetime

Let $\rho(\lambda)$ be a full-rank family on a finite tensor product. Umegaki relative entropy is differentiable, its BKM Hessian is positive, and the entanglement first law holds about every faithful reference state. Choose no map to a manifold, no causal completion, and no massless spin-two mode. The microscopic premise theorem 4.1 holds, while theorem 4.2 does not. No spacetime tensor equation is even well typed. A gapped spin model or a topological phase supplies a physically familiar realization of this logical case.

13.2 Positive information geometry without causal reconstruction

An injective probe can pull a Petz metric back to a positive Riemannian metric h on a parameter manifold. Without a clock, causal order, or another signature-changing rule, h is not a Lorentzian metric. Declaring one coordinate to be time by notation does not produce a natural causal structure. This countermodel violates only the causal part of theorem 4.2 while leaving distinguishability geometry intact.

13.3 A preferred chart without diffeomorphism redundancy

Suppose a reconstruction defines tensor components relative to a fixed coordinate function that is not transformed with the microscopic relabeling. Then generally

$$\mathfrak{R}_\mu(\alpha_f q) \neq f^*\mathfrak{R}_\mu(q). \tag{53}$$

One may still write a smooth metric array and a local action in that chart, but the claimed microscopic redundancy does not descend to $\mathrm{Diff}(M)$. The Ward and quotient steps of the theorem are unavailable.

13.4 A conformal tangent that tests only the trace

Let $g_{\mu\nu}(\sigma) = e^{2\sigma}\bar{g}_{\mu\nu}$. Then $\delta g^{\mu\nu} = -2\delta\sigma g^{\mu\nu}$, so

$$\delta_\sigma\widehat{\Gamma} = -\frac{1}{8\pi G_{\mathrm{N}}}\int_M \mathrm{d}^d x\sqrt{-g}\,\mathcal{C}^\mu{}_\mu\delta\sigma. \tag{54}$$

Stationarity gives only $\mathcal{C}^\mu{}_\mu = 0$. An arbitrary nonzero traceless defect remains invisible. This is the concrete obstruction in theorem 9.3.

13.5 A stationary identity is absent

Fix any weakly curved metric that is off shell for the chosen effective action and let the microscopic reconstruction be constant in its allowed state directions. Locality, covariance, and a controlled operator basis may still be present, but theorem 4.6 fails because the pulled-back action need not be extremal. The EFT classification tells us the form of the residual; it does not set that residual to zero.

13.6 A nonlocal action outside a finite-jet theorem

An all-to-all reconstruction kernel or the massless form factor in equation (24) makes the metric Euler derivative depend on fields away from a point. The full nonlocal equation can remain covariant and Ward compatible. Replacing it by finitely many local coefficients changes the functional and can create spurious solutions. The local part of theorem 9.1 survives only with H^{nl} separately retained.

13.7 An additional light field

Consider

$$\Gamma[g,\varphi] = \int_M \mathrm{d}^d x \sqrt{-g}\left[\frac{1}{2}F(\varphi)R - \frac{1}{2}(\nabla\varphi)^2 - V(\varphi)\right] + \Gamma_{\mathrm{m}}. \tag{55}$$

When φ is light, its equation and nonminimal metric contribution remain at leading order. A pure Einstein equation with constant G_{N} is not the complete law unless φ is stabilized, decoupled, or included in the retained conclusion. Multiple-metric and vector-tensor theories give the same lesson for other spectra.

13.8 An uncontrolled higher-curvature coefficient

For

$$\Gamma = \frac{1}{16\pi G_{\mathrm{N}}}\int \sqrt{-g}\,R + a\int \sqrt{-g}\,R^2, \tag{56}$$

the ratio of the four-derivative tensor to the Einstein tensor on scale L is of order $16\pi G_{\mathrm{N}}|a|/L^2$. Choosing $|a| \sim L^2/(16\pi G_{\mathrm{N}})$ makes the nominal correction leading even when L is macroscopic. Scale separation without coefficient control is therefore insufficient.

If the higher-derivative action is treated nonperturbatively, additional poles may also enter, as quadratic gravity demonstrates [10]. The EFT theorem instead treats such terms perturbatively below their new-state scale unless those states are explicitly retained.

13.9 Dimension and topology outside the stated domain

In $d = 2$, the Einstein–Hilbert integral is topological and the Einstein tensor vanishes identically. In $d = 3$, pure Einstein gravity has no local bulk graviton polarization. In $d \geq 5$, higher Lovelock tensors can be nontrivial. These cases do not refute theorem 9.1; they show why theorem 4.8 fixes the dimension and intended physical statement.

13.10 An anomaly or unbalanced source

If chiral matter has an uncanceled diffeomorphism anomaly,

$$\nabla^\mu T_{\mu\nu} = \mathcal{A}_\nu, \tag{57}$$

a divergence-free pure-metric Euler tensor cannot equal the stress tensor unless inflow or additional degrees of freedom supply the missing variation [11]. Likewise, a spacetime-dependent external source can exert a Ward force. Formal covariance with the source transformed as a spurion exposes the force; it does not make the source equation vanish.

13.11 A boundary with physical flux

Let matter energy leave a finite region through a timelike boundary. The local bulk identity can hold while the integrated charge changes by the surface flux. Setting the boundary term to zero without reflective data is incorrect. The Einstein–Hilbert Dirichlet problem on a smooth non-null boundary uses the Gibbons–Hawking–York completion [12, 13]; null boundaries, corners, and higher-curvature theories require their own completions.

13.12 First law without an entropy-gravity dictionary

The finite faithful system in the first countermodel obeys $\delta S = \delta\langle K\rangle$ and $D \geq 0$. Choose no area functional, stress dictionary, or complete family of spacetime regions. The entanglement identities remain true and no gravitational constraint follows. Thus the middle column of table 2 contains substantive input.

13.13 A beta flow without a variational relation

On $\mathbb{R}^2$ let

$$\beta(x, y) = (-y, x). \tag{58}$$

The trajectories are circles and $\beta \cdot \nabla(x^2 + y^2) = 0$. The flow is not strict descent of the radial potential and its zeros need not coincide with the stationary set of a chosen effective functional. A smooth scale flow therefore does not imply equation (45). A singular mobility gives another failure: a nonzero defect can lie in its kernel and appear as a fixed point.

13.14 A degenerate or indefinite response

Before quotienting diffeomorphisms, the action Hessian has gauge zero modes. Moduli can leave physical zero modes, and a ghostlike kinetic term violates positivity. The Lorentzian conformal sector also obstructs a naive positive metric on the full space of metric components. In any of these cases, a positive microscopic Fisher tensor cannot simply be identified with the raw gravitational Hessian.

Proposition 13.1 (No theorem for all microscopic systems)**.** *The standard axioms of finite-dimensional quantum mechanics, the existence of a faithful state family, monotonicity of a quantum information metric, and the relative-entropy first law do not imply the existence of an admissible record in theorem 4.10.*

Proof. The faithful finite system with no map to a manifold satisfies all listed quantum-information premises and fails theorem 4.2. Adding an arbitrary parameter-space Riemannian metric does not supply causal completion, diffeomorphism redundancy, a local effective action, or a spin-two spectrum. Hence the asserted implication has a countermodel. □

14 Representative records

14.1 Weakly curved four-dimensional metric EFT

Suppose a model-specific reconstruction has verified theorems 4.1 to 4.4. Let its four-dimensional infrared functional contain one metric, anomaly-free matter, and the local action equation (20), with $L \gg \ell_*$ and bounded a_i. If the pulled-back action is stationary and the boundary problem is well posed, theorem 9.1 gives the full equation. At four-derivative order,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + 16\pi G_{\mathrm{N}} \sum_{i=1}^{3} a_i H^{(i)}_{\mu\nu} + H^{\mathrm{nl}}_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu} + \mathcal{O}(\partial^6). \tag{59}$$

The theorem does not compute the a_i, and it does not say that the reconstruction assumptions hold merely because this action can be written.

14.2 A holographic perturbative intersection

For a semiclassical AdS code subspace with the RT or generalized-entropy and stress dictionaries, first laws for all boundary balls impose the linearized bulk equation. When the same bulk reconstruction also supplies the local field-space and EFT hypotheses of this paper, the entanglement conclusion is the linearization of equation (35). Boundary relative-entropy Fisher information can equal bulk canonical energy on this shared domain [19, 20]. This is a concrete intersection of routes, not evidence that their dictionaries hold in a generic lattice model.

14.3 Flat vacuum and higher-curvature terms

For $T_{\mu\nu} = 0$, $\Lambda = 0$, and a flat metric, all local curvature tensors and the representative curvature-squared nonlocal terms vanish. The record is defect free if the remaining retained equations and boundary data hold. This elementary example checks the normalization but not tangent completeness or microscopic reconstruction.

14.4 A finite projected record

Let the reconstruction return only homogeneous isotropic metrics with a lapse and scale factor. Stationarity gives the minisuperspace constraint and evolution equation. These are valid projections of equation (35); they do not test inhomogeneous tensor components. The proper conclusion is theorem 9.3, not the full local theorem.

15 What the theorem establishes

It is useful to separate five statements that are sometimes compressed into one slogan.

(1) A specified quantum state family carries a specified information metric. This is a state-space statement.

(2) A model-specific interface returns a local Lorentzian metric and a physical tangent map. This is a reconstruction statement.

(3) A diffeomorphism-invariant effective functional defines a defect and a Ward identity. This is an infrared field-theory statement.

(4) Stationarity plus tangent completeness sets the defect to zero. This is a variational implication.

(5) A one-metric controlled derivative expansion makes the leading local equation Einsteinian. This is an EFT universality statement.

Only the last three steps are derived once their antecedents are present. The first step does not imply the second. The theorem is therefore compatible with a world containing many quantum systems that never reconstruct spacetime.

The result also does not prove:

(a) a microscopic formula for G_{N} or Λ;

(b) a solution to the cosmological constant problem;

(c) nonlinear gravitational dynamics from a first-order entanglement law;

(d) locality of an arbitrary geometric coarse-graining;

(e) equality of a generic metric beta functional with the defect;

(f) positivity of the Lorentzian gravitational Hessian;

(g) uniqueness of the microscopic description or reconstruction map;

(h) a physical relaxation law for a nonzero defect.

The theorem does provide a disciplined target for microscopic model building. A proposal can report which clauses of theorem 4.10 it proves, computes, assumes, or falsifies in ordinary scientific prose. Failure of one clause locates the missing bridge rather than being hidden behind the phrase "emergent geometry."

16 Finite executable model

The accompanying Haskell modules implement a finite dependency algebra, an exact rational correction budget, and small physical residual vectors. The core module defines the required hypothesis set and admits a record only when every required item is present. It also separates three optional route tags from the core antecedent. The proof module checks remove-one-hypothesis witnesses, the exact zero-correction limit, a projected-tangent counterexample, and invariance of a zero residual under an invertible finite coordinate change.

The property module uses QuickCheck with fixed seeds. Its tests include:

(i) every complete required set is admitted;

(ii) deleting a required hypothesis makes the finite record inadmissible;

(iii) optional route tags do not repair a missing core hypothesis;

(iv) nonnegative correction budgets give monotone bounds;

(v) componentwise residual addition obeys the ℓ_1 triangle bound;

(vi) a zero budget gives an exact leading equation;

(vii) permuting correction entries does not change the total budget.

The main program runs the deterministic proof checks and then the seeded QuickCheck suite. Compilation uses warnings as errors.

These tests are deliberately modest. A finite list can validate that the logical dependency code matches its specification. It cannot establish Fréchet differentiability, a continuum Ward identity, density of an infinite-dimensional tangent, or the spectrum of a microscopic quantum system. Those are hypotheses and analytic obligations in the paper.

17 Discussion

The reconstruction consistency theorem is best read as a commutative square between two kinds of variation. Microscopic perturbations move through the reconstruction tangent to infrared fields. Infrared functional variation produces Euler covectors. The chain rule pairs the two. If the tangent is complete on the physical quotient, vanishing of the pulled-back covector is equivalent to vanishing of the metric defect. This functional-analytic step, not the notation used for the microscopic coordinates, is what permits a local tensor equation.

Diffeomorphism invariance plays a different role. It ensures that redundant metric directions do not provide independent tests and that the defect obeys the appropriate Noether–Ward identity once other equations are imposed. It does not choose a stationary point. The conserved tensor $\lambda g_{\mu\nu}$ makes this distinction elementary.

The effective expansion then explains why the Einstein tensor appears. The argument is not that "information" has the algebraic form of curvature. It is that a local, parity-even, diffeomorphism-redundant theory with one nondegenerate metric spin-two field has a sharply restricted two-derivative operator basis. Information geometry enters upstream through the supplied reconstruction and, in special dictionaries, through a quadratic response.

This reading also clarifies the word universality. Universality of operator form is common in low-energy physics even when UV realizations and couplings differ. The massless spectrum and symmetries determine the allowed leading operators. The scale hierarchy orders the rest. The present theorem applies that logic after, not before, a model-specific reconstruction has produced the appropriate infrared field.

The nonlocal term is an important guardrail. Integrating out a heavy field gives an analytic expansion under standard decoupling assumptions [9]. Retained massless propagation does not. Running local coefficients can cancel the explicit renormalization scale of a logarithmic form factor without making its metric variation vanish. Thus Callan–Symanzik scale independence and reconstruction stationarity remain separate equations.

Finally, the three auxiliary routes are useful precisely because they are not identical. Entanglement gives region-dependent first-law constraints. Beta-functionals give fixed-point statements on a scale bundle. Stiffness is a quadratic response coefficient. Where a concrete model supplies all their dictionaries, they can meet on the same defect and reinforce the consistency picture. Where it does not, analogy should not be promoted to implication.

18 Conclusion

An information-geometric reconstruction can support an Einstein-form infrared equation, but the conclusion begins with an admissible reconstruction record. The record names a regular microscopic domain and metric, a smooth local Lorentzian output, natural diffeomorphism redundancy, a complete physical tangent, a stationary and well-posed effective functional, Ward and boundary compatibility, one nondegenerate metric spin-two mode, and a controlled scale window.

Under those hypotheses, the proof is short once its modules are visible. The functional chain rule pulls the defect covector back to reconstruction space. Stationarity annihilates that pullback. Ward compatibility removes gauge directions. Tangent completeness separates the covector and makes the full defect vanish. The infrared operator classification then gives

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + H^{\mathrm{loc}}_{\mu\nu} + H^{\mathrm{nl}}_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}. \tag{60}$$

The Einstein equation is the leading local zero- and two-derivative part, with the correction estimate equation (42) stating the precise sense in which that limit is approached.

The same theorem also says what does not follow. A monotone quantum metric does not supply a causal manifold, a gauge principle, an effective action, or a massless spin-two spectrum. The entanglement first law does not supply an entropy-area or stress dictionary. A metric flow does not supply a gradient relation. A quadratic gravitational kernel does not equal microscopic Fisher information without a response dictionary. Therefore no conclusion over all microscopic systems is available.

This narrower result is still useful. It turns a broad emergence claim into a finite family of mathematical and physical tests, identifies the universal leading operator inside the passing class, and keeps every correction or obstruction in its proper place.

A Hypothesis trace through the series

The proof uses earlier results through the following dependency trace. The table records logical use, not historical priority.

Source	Imported object	Use here
Part I	Specified Petz metric, local datum, reconstruction interface, causal completion, variation map	Fixes the microscopic domain and prevents a positive information metric from being mistaken for Lorentzian geometry
Part II	Regularity, locality, redundancy, tangent, Ward, spectrum, dimension, and scale hypotheses	Defines the admissible infrared reconstruction class and remove-one-hypothesis tests

Source	Imported object	Use here
Part III	Inverse-metric normalization, reconstruction stationarity, local operator variation, nonlocal remainder	Supplies theorems 7.1 and 8.1 and the expanded defect
Part IV	Holographic and causal-diamond dictionaries; perturbative order separation	Supplies only theorem 11.1, not nonlinear core stationarity
Part V	Natural defect, off-shell Ward identity, boundary/source/anomaly terms, projected stationarity	Supplies theorem 7.4 and the distinction between conservation and vanishing
Part VI	Comparison connection, local reconstruction RG, physical beta, variational mobility	Supplies theorem 11.2 under its additional hypotheses
Part VII	Gauge-quotiented quadratic metric response and its information dictionary	Supplies the interpretation in theorem 11.3; it is not needed to set the first variation to zero

The core implication could be stated without Parts IV, VI, and VII. Including them in the series theorem identifies three independent ways to test a stationarity premise or interpret a coupling. Their dictionaries are kept visible so that agreement of conclusions is not mistaken for equivalence of premises.

B Representative curvature-squared tensors

For the curvature conventions used in the series, define

$$\delta \int_M \mathrm{d}^d x\sqrt{-g}\, R^2 = \int_M \mathrm{d}^d x\sqrt{-g}\, H^{(1)}_{\mu\nu}\delta g^{\mu\nu}, \tag{61}$$

$$\delta \int_M \mathrm{d}^d x\sqrt{-g}\, R_{\alpha\beta}R^{\alpha\beta} = \int_M \mathrm{d}^d x\sqrt{-g}\, H^{(2)}_{\mu\nu}\delta g^{\mu\nu}, \tag{62}$$

after the relevant boundary terms are removed. Direct variation gives

$$H^{(1)}_{\mu\nu} = 2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2 + 2(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R, \tag{63}$$

$$\begin{aligned} H^{(2)}_{\mu\nu} &= 2R_{\mu\alpha\nu\beta}R^{\alpha\beta} - \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \Box R_{\mu\nu} \\ &\quad + \frac{1}{2}g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R. \end{aligned} \tag{64}$$

The quadratic Euler density is

$$\mathcal{X}_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \tag{65}$$

Its bulk Euler tensor is the Lanczos tensor

$$\begin{aligned} H^{(\mathrm{GB})}_{\mu\nu} = 2\big(&RR_{\mu\nu} - 2R_{\mu\alpha}R_\nu{}^\alpha - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} \\ &+ R_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}\big) - \frac{1}{2}g_{\mu\nu}\mathcal{X}_4. \end{aligned} \tag{66}$$

For integer $d \le 4$ (without dimensional continuation), this Lanczos tensor vanishes identically; in $d = 4$ its integral is the topological Euler characteristic up to the standard boundary completion. The Riemann-squared variation obeys

$$H^{(3)}_{\mu\nu} = H^{(\mathrm{GB})}_{\mu\nu} + 4H^{(2)}_{\mu\nu} - H^{(1)}_{\mu\nu}. \tag{67}$$

Each tensor is divergence compatible as the Euler derivative of a diffeomorphism-invariant scalar functional. This provides a useful sign check and another demonstration that Ward compatibility does not force a tensor to vanish.

The normalization in equation (59) is

$$H^{\mathrm{loc}}_{\mu\nu} = 16\pi G_{\mathrm{N}} \left(a_1 H^{(1)}_{\mu\nu} + a_2 H^{(2)}_{\mu\nu} + a_3 H^{(3)}_{\mu\nu} + \cdots \right). \tag{68}$$

The factor $16\pi G_{\mathrm{N}}$ is required because the a_i terms in equation (20) were written outside the Einstein–Hilbert prefactor.

C Finite dependency algebra

Let the finite hypothesis universe be

$$\mathsf{H} = \{M, R, N, T, V, S, W, Q, E\}, \tag{69}$$

standing for microscopic regularity, reconstruction, natural locality, tangent completeness, variational well-posedness, stationarity, Ward compatibility, spectrum, and expansion control. Define

$$\mathsf{adm}(X) \quad \Longleftrightarrow \quad \mathsf{H} \subseteq X. \tag{70}$$

Then for every $h \in \mathsf{H}$,

$$\mathsf{adm}(\mathsf{H}) = \text{true}, \qquad \mathsf{adm}(\mathsf{H} \setminus \{h\}) = \text{false}. \tag{71}$$

This is a check of the declared dependency algebra, not a proof that the physical hypotheses are independent in every model class.

Let a finite correction budget be

$$b = (b_{\mathrm{loc}}, b_{\mathrm{nl}}, b_{\mathrm{light}}), \qquad b_i \geq 0, \tag{72}$$

with total $|b|_1 = \sum_i b_i$. For a finite residual vector r and correction vectors u_i satisfying $r + \sum_i u_i = 0$, the ordinary triangle inequality gives

$$\|r\|_1 \leq \sum_i \|u_i\|_1. \tag{73}$$

This is the discrete analogue of equation (42). The Haskell code uses exact rational arithmetic so that no floating-point tolerance obscures the logical checks.

D Verification boundaries

The continuum theorem uses several analytic statements that are not reduced to the finite program:

(1) existence and smoothness of the reconstruction map;

(2) locality estimates in a continuum topology;

(3) density of the reconstruction tangent modulo gauge;

(4) well-posed functional derivatives and boundary completions;

(5) quantum anomaly cancellation and source balance;

(6) the actual pole content and sign of the infrared spectrum;

(7) coefficient bounds and nonlocal form factors in a microscopic model.

The executable model checks algebra after these items are represented as Boolean or rational inputs. Passing those checks cannot turn an unchecked physical hypothesis into an empirical fact.

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