Newton's Constant as the Stiffness of the Infrared Metric

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Abstract

The coefficient of the Einstein–Hilbert term is often described as the rigidity of spacetime. This paper makes that statement precise and records its limits. We expand a gravitational effective action around a solution, remove diffeomorphism directions by gauge fixing or passage to the physical quotient, and define stiffness as the quadratic inverse propagator of the remaining metric modes. For a dimensionless metric perturbation about flat space, the transverse-traceless sector has Euclidean Hessian $\mathcal{K}_s(k) = k^2/(32\pi G_d)$ in the conventions used here. Thus the low-momentum metric kernel scales as k^2/G_d, while the response to a transverse-traceless source scales as G_d/k^2. On an Einstein background the same conclusion holds for the principal symbol, with curvature and boundary-condition dependent lower-derivative terms. The statement is not a positivity claim about the unreduced Euclidean gravitational action: gauge zero modes, the conformal-factor direction, moduli, and unstable modes must be treated separately.

We then compare this effective-field-theory stiffness with information geometry. A microscopic relative-entropy Hessian equals bulk canonical energy only when an explicit holographic dictionary supplies a code subspace, a state-to-field map, the appropriate boundary conditions, and a controlled gauge. Under those hypotheses the Einstein symplectic form carries its characteristic factor of $1/G_d$. Outside such a dictionary, calling a microscopic Fisher metric the origin of Newton's constant is an interpretation, not a derived equality. Induced-gravity calculations likewise show how quantum fields renormalize the coefficient of curvature, but the answer depends on the spectrum, masses, couplings, regulator or ultraviolet completion, bare counterterm, and renormalization condition. Consequently no numerical value of Newton's constant follows from stiffness, information geometry, or induced gravity alone. The robust result is narrower: after normalization and gauge data are fixed, $1/G_d$ is the coefficient that measures the quadratic resistance of the infrared metric to physical deformation in its two-derivative regime.

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1 Introduction

Newton's constant plays two roles in the Einstein equation. It multiplies the source in the response equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_d T_{\mu\nu}, \tag{1}$$

and its inverse multiplies the gravitational action

$$S_{\rm EH}[g] = \frac{1}{16\pi G_d}\int_M \mathrm{d}^d x\,\sqrt{-g}\,(R-2\Lambda) + S_{\partial M}. \tag{2}$$

The first form says that a larger G_d produces a larger response to a fixed stress tensor. The second says that a larger $1/G_d$ assigns a larger quadratic action to a fixed metric deformation. These are the susceptibility and stiffness descriptions of the same linearized operator.

The analogy is useful, but the raw second derivative of (2) is not yet a stiffness matrix. Diffeomorphism invariance makes the Hessian degenerate. The Euclidean conformal direction has the wrong sign. A Lorentzian action is not a positive quadratic form. Constant and on-shell massless modes can lie in the kernel. The physical content appears only after a background, boundary conditions, normalization, and a physical mode space have been specified. In that setting, the principal symbol on transverse-traceless polarizations is proportional to k^2/G_d. This is the precise statement developed below.

There is a second possible meaning of stiffness. A family of quantum states has a positive Hessian of relative entropy. In favorable circumstances this information metric can be mapped to a gravitational canonical energy. The best-established example is a holographic conformal field theory, a ball-shaped boundary region in the vacuum, and perturbations in a semiclassical code subspace. There the relevant relative-entropy Hessian agrees with bulk canonical energy to the appropriate order [10, 7]. The equality is powerful because the bulk symplectic form knows the normalization of the gravitational action. It is also conditional: without the holographic state-to-field map, the equality is not an identity between an arbitrary quantum Fisher metric and the Einstein Hessian.

Induced gravity provides a third route to the same coefficient. Integrating out quantum fields produces a term proportional to $\int\sqrt{g}R$ in the effective action. Sakharov's elasticity analogy and later induced-gravity calculations make the stiffness language especially natural [13, 1, 15, 14]. Yet the induced coefficient is not universal. It depends on the field content, nonminimal couplings, masses, threshold prescription, regulator, and the bare gravitational counterterm. An induced contribution explains how inverse Newton couplings are generated and run; by itself it does not predict the measured value.

The central claim of this paper is deliberately limited.

Proposition 1.1 (Quadratic stiffness claim)**.** *Let an infrared gravitational effective action contain* (2)*, let* $\bar{g}$ *solve its background equations, and fix the normalization of the metric by its standard line element and matter coupling. After quotienting proper diffeomorphisms or imposing a complete gauge, the two-derivative principal symbol of the quadratic operator on a physical transverse-traceless polarization is proportional to*

$$\frac{k^2}{G_d}. \tag{3}$$

For the flat-space conventions of section 4, the exact Hessian eigenvalue is $k^2/(32\pi G_d)$*. This coefficient justifies calling* $1/G_d$ *the stiffness of the infrared metric, within the specified physical sector and derivative expansion.*

The proposition neither derives a microscopic origin for G_d nor fixes a number. It interprets a coefficient in a specified effective action. A microscopic derivation requires a map whose pullback identifies the microscopic quadratic form with the gravitational one. A numerical prediction additionally requires enough microscopic data and a matching or renormalization condition to determine the coefficient.

The paper is organized as follows. Section 2 fixes the effective-action and normalization conventions. Section 3 constructs the physical quadratic form. Section 4 derives the flat-space kernel and source response; section 5 gives the curved-background generalization. Sections 6 and 7 treat zero modes, positivity, units, and field normalization. Section 8 states the conditional holographic information identity, while section 9 contrasts it with an ordinary flat-space graviton. Section 10 discusses induced gravity, and section 11 states the domain of the two-derivative approximation. Section 12 proves the underdetermination of a numerical Newton constant. The appendices audit the projector algebra, the Gaussian analogy, and the executable rational model accompanying the paper.

2 Effective action, background, and normalization

2.1 Infrared data

Let M be a smooth d-dimensional spacetime. Unless otherwise stated, the Lorentzian metric has signature $(- + \cdots +)$, and natural units $c = \hbar = 1$ are used. The infrared effective action at a scale μ is organized as

$$\Gamma_{\mathrm{IR}}[g, \psi; \mu] = S_{\mathrm{EH}}[g; G_d(\mu), \Lambda(\mu)] + S_{\mathrm{m}}[g, \psi; \mu] + \Gamma_{\mathrm{hd}}[g, \psi; \mu] + \Gamma_{\mathrm{nl}}[g, \psi; \mu]. \tag{4}$$

Here Γ_{hd} contains local higher-derivative operators and Γ_{nl} contains any nonlocal terms retained in the effective description. Boundary terms and boundary conditions are part of the action data. For Dirichlet Einstein gravity, $S_{\partial M}$ in (2) includes the Gibbons–Hawking–York term and the counterterms appropriate to the asymptotics.

Choose a background $(\bar{g}, \bar{\psi})$ satisfying the equations derived from the retained action. A metric variation is written

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{5}$$

The perturbation $h_{\mu\nu}$ is dimensionless because coordinates carry length and the line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ carries length squared. This convention is important: rescaling h rescales every entry of the Hessian. The standard metric normalization is fixed physically by the universal matter coupling and by the geometric meaning of proper time and distance.

The expansion about the background is

$$\Gamma_{\mathrm{IR}}[\bar{g}+h, \bar{\psi}+\delta\psi] = \Gamma_{\mathrm{IR}}[\bar{g}, \bar{\psi}] + \Gamma^{(1)}_{\mathrm{IR}}[h, \delta\psi] + \Gamma^{(2)}_{\mathrm{IR}}[h, \delta\psi] + O(\delta\Phi^3), \tag{6}$$

where $\delta\Phi = (h, \delta\psi)$. On a solution, the first variation vanishes for perturbations satisfying the prescribed boundary conditions. The second-order term may be written

$$\Gamma^{(2)}_{\mathrm{IR}} = \frac{1}{2}\langle \delta\Phi, \mathcal{K}_{\bar{\Phi}}\delta\Phi\rangle. \tag{7}$$

The operator $\mathcal{K}_{\bar{\Phi}}$ is the Hessian only after its domain and boundary conditions have been fixed. If metric and matter fluctuations mix, it is a block operator rather than a scalar metric kernel.

2.2 What stiffness means

In ordinary mechanics, a quadratic energy

$$E(q) = \frac{1}{2}q^A K_{AB} q^B - J_A q^A \tag{8}$$

has linear response $q = K^{-1}J$ on the invertible subspace. Increasing K reduces the response to a fixed source. We use the same word for a field theory, with two qualifications.

First, a massless local field has a derivative stiffness. Its inverse propagator vanishes at $k = 0$, even when its gradient coefficient is large. The coefficient of k^2, rather than the eigenvalue at $k = 0$, is the stiffness parameter. Second, a gauge theory has redundant directions. The physical stiffness is the quadratic form induced on the gauge quotient or, equivalently when the gauge is complete, on a gauge-fixed representative space.

Definition 2.1 (Infrared metric stiffness)**.** Fix a background solution, boundary data, a normalization of $h_{\mu\nu}$, and a physical perturbation space obtained after removing proper gauge directions. The infrared metric stiffness is the coefficient of the leading two-derivative part of the quadratic inverse propagator on that space. For an Einstein–Hilbert action it is proportional to $1/G_d$.

This is an effective-field-theory definition. It does not suppose that the metric is literally an elastic medium, that spacetime is embedded in a higher-dimensional material, or that all physical modes have positive energy on every background.

2.3 The variational source convention

The stress tensor convention used below is

$$\delta S_{\mathrm{m}} = -\frac{1}{2}\int_M \mathrm{d}^d x\sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu}. \tag{9}$$

Since $\delta g^{\mu\nu} = -h^{\mu\nu} + O(h^2)$, the linear Lorentzian matter coupling is

$$S^{(1)}_{\mathrm{m}} = \frac{1}{2}\int_M \mathrm{d}^d x\, h_{\mu\nu} T^{\mu\nu} \tag{10}$$

in flat space. A Euclidean generating functional may use the opposite sign after Wick rotation. The sign of the response then follows that convention; its magnitude and its G_d/k^2 scaling do not.

3 The physical Hessian and the diffeomorphism quotient

3.1 Gauge degeneracy

An infinitesimal diffeomorphism generated by a vector field ξ^μ acts on the background split as

$$h_{\mu\nu} \longmapsto h_{\mu\nu} + 2\bar{\nabla}_{(\mu}\xi_{\nu)}. \tag{11}$$

If ξ is a proper gauge transformation, meaning that it preserves the boundary conditions and has vanishing associated boundary charge, then it does not change the physical state. On a background solution, diffeomorphism invariance implies

$$\mathcal{K}_{\bar{g}}\big(2\bar{\nabla}_{(\mu}\xi_{\nu)}\big) = 0 \tag{12}$$

up to boundary terms and coupled equations that must be included when other background fields transform. Thus the unreduced Hessian is singular by construction.

The adjective proper matters. A diffeomorphism that changes asymptotic data or carries a nonzero surface charge can connect distinct physical states. Such a transformation cannot simply be divided out. Likewise, an exact Killing vector gives a reducibility parameter: its metric variation vanishes, but its presence can produce zero modes in the gauge-fixing and ghost operators that require separate treatment.

Let $\mathcal{V}_{\mathrm{adm}}$ denote admissible linearized perturbations and $\mathcal{V}_{\mathrm{gauge}}$ the subspace generated by proper diffeomorphisms. The physical tangent space is

$$\mathcal{V}_{\mathrm{phys}} = \mathcal{V}_{\mathrm{adm}}/\mathcal{V}_{\mathrm{gauge}}. \tag{13}$$

The quadratic form descends to this quotient when the variational principle and boundary terms are well defined.

Proposition 3.1 (Descent of the quadratic form)**.** *Suppose the background solves the full retained equations, proper diffeomorphisms preserve the admissible boundary data, and no diffeomorphism anomaly is present. Then the on-shell Hessian pairs a proper gauge direction with every admissible perturbation to zero. Consequently it defines a bilinear form on* $\mathcal{V}_{\mathrm{phys}}$*.*

Proof. Let $\delta_\xi\Phi = \mathcal{L}_\xi\bar{\Phi}$ denote the transformation of every background field. Gauge invariance gives $\delta_\xi\Gamma_{\mathrm{IR}}[\Phi] = 0$ for proper ξ. Differentiate this identity once with respect to an arbitrary admissible perturbation. The term from the variation of the Euler derivative vanishes because the background solves the equations, while the remaining term is the Hessian pairing $\Gamma^{(2)}_{\mathrm{IR}}(\delta\Phi, \delta_\xi\Phi)$. Boundary terms vanish by the definition of proper gauge transformations and the assumed variational principle. Therefore the pairing depends only on quotient classes. □

3.2 Gauge fixing and equivalence

For perturbative calculations it is often more convenient to select representatives. In flat space define the trace reverse

$$\bar{h}^{\mathrm{tr}}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \qquad h := \eta^{\mu\nu}h_{\mu\nu}, \tag{14}$$

where the superscript prevents confusion with the background metric. The de Donder condition is

$$\partial^\mu \bar{h}^{\mathrm{tr}}_{\mu\nu} = 0. \tag{15}$$

With residual gauge freedom and the vacuum constraints imposed, a propagating mode can be represented by its spatial transverse-traceless components,

$$h^{\mathrm{TT}}_{0\mu} = 0, \qquad \partial_i h^{\mathrm{TT}}_{ij} = 0, \qquad \delta^{ij} h^{\mathrm{TT}}_{ij} = 0. \tag{16}$$

Gauge fixing adds an operator along redundant directions and introduces a Faddeev–Popov determinant. Gauge-parameter dependence in intermediate propagators does not alter the coefficient on a physical polarization. A complete gauge slice with nonsingular Faddeev–Popov operator is locally isomorphic to the quotient. If residual gauge transformations, Killing vectors, Gribov issues, or boundary edge modes remain, that equivalence needs additional qualifications. We will use the transverse-traceless slice only where these assumptions are explicit.

3.3 Counting local polarizations

In d-dimensional Minkowski space, a massless graviton has

$$N_{\mathrm{pol}}(d) = \frac{d(d-3)}{2}, \qquad d \geq 4, \tag{17}$$

local helicity states. This follows either from the little-group symmetric traceless tensor or from the constraints and gauge quotient. In $d = 4$ it gives the familiar two polarizations. In $d = 3$ pure Einstein gravity has no local graviton, even though it can have boundary gravitons and global degrees of freedom. In $d = 2$ the Einstein–Hilbert term is topological. Therefore the statement "$1/G_d$ is graviton stiffness" must be understood through the action or boundary/global sectors in these low dimensions rather than a nonexistent local transverse-traceless particle.

4 Flat-space quadratic kernel

4.1 Fierz–Pauli expansion

Set $\Lambda = 0$, take $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, and neglect higher-derivative terms. Up to a boundary term, the quadratic expansion of the Einstein–Hilbert action is the massless Fierz–Pauli action [5, 3]:

$$\begin{aligned} S_{\mathrm{FP}}[h] = \frac{1}{64\pi G_d} \int \mathrm{d}^d x \, \big(&-\partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + 2\partial_\mu h^{\mu\nu} \partial^\lambda h_{\lambda\nu} \\ &- 2\partial_\mu h^{\mu\nu} \partial_\nu h + \partial_\lambda h \partial^\lambda h \big), \end{aligned} \tag{18}$$

using the curvature-sign convention for which the transverse-traceless Hamiltonian is positive in signature $(- + \cdots +)$ when $G_d > 0$. Equivalent forms differ by integration by parts; reversing the Riemann-tensor convention requires the corresponding sign change in the Einstein action.

Restricting to equation (16) gives

$$S^{(2)}_{\mathrm{TT}} = \frac{1}{64\pi G_d} \int \mathrm{d}^d x \, \left[(\partial_t h^{\mathrm{TT}}_{ij})^2 - (\partial_k h^{\mathrm{TT}}_{ij})^2 \right]. \tag{19}$$

This is the kinetic term for the physical polarizations. The same coefficient appears in the symplectic form and canonical energy, subject to the standard normalization of the perturbation.

After Wick rotation, choose normalized transverse-traceless polarization tensors $e^s_{ij}(k)$ satisfying

$$e^s_{ij} e^{s'\,ij} = \delta^{ss'}, \qquad k^i e^s_{ij} = 0, \qquad \delta^{ij} e^s_{ij} = 0, \tag{20}$$

and expand

$$h_{ij}^{\mathrm{TT}}(k) = \sum_s q_s(k) e_{ij}^s(k). \tag{21}$$

The Euclidean quadratic action is

$$I_{\mathrm{TT}}^{(2)} = \frac{1}{64\pi G_d} \sum_s \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, k^2 \, |q_s(k)|^2. \tag{22}$$

Writing this as $I^{(2)} = \frac{1}{2}\sum_s \int q_s^* \mathcal{K}_s q_s$ gives

$$\boxed{\mathcal{K}_s(k) = \frac{k^2}{32\pi G_d}.} \tag{23}$$

This exact factor depends on the normalization (20) and on taking $g = \eta + h$. The invariant content for the present argument is the scaling k^2/G_d.

Proposition 4.1 (Flat physical kernel)**.** *For $d \geq 4$, $G_d > 0$, nonzero Euclidean momentum, and the normalization above, the Einstein–Hilbert Hessian restricted to each flat-space physical transverse-traceless polarization is positive and equals $k^2/(32\pi G_d)$. It is linear in k^2 and inverse-linear in G_d.*

Proof. Substitute equation (21) into equation (22). Orthogonality removes the cross terms. Comparing the coefficient of $|q_s|^2$ with the Hessian convention $I^{(2)} = \frac{1}{2} q^* \mathcal{K} q$ yields equation (23). Positivity follows from $k^2 > 0$ and $G_d > 0$. □

4.2 Response to a source

Project the source coupling equation (10) onto the same polarization. With the Fourier reality conditions understood, use the Euclidean convention

$$I[q;T] = \frac{1}{2} q^* \mathcal{K} q - \frac{1}{2} q^* T_{\mathrm{TT}}. \tag{24}$$

Stationarity gives $\mathcal{K} q = T_{\mathrm{TT}}/2$, and hence

$$q_s(k) = \frac{16\pi G_d}{k^2} \, T_s^{\mathrm{TT}}(k) \tag{25}$$

when T_s^{TT} denotes the projection of the stress tensor in (10). The latter is the standard normalization from the linearized Einstein equation

$$\Box \bar{h}_{\mu\nu}^{\mathrm{tr}} = -16\pi G_d T_{\mu\nu} \tag{26}$$

in de Donder gauge. Fourier and Wick-rotation sign choices can change the sign of equation (25); they do not change its magnitude.

The response displays the inverse relationship:

$$\text{stiffness} \sim \frac{k^2}{G_d}, \qquad \text{susceptibility} \sim \frac{G_d}{k^2}. \tag{27}$$

It also shows why the zero mode must be excluded before inverting the operator. A spatially homogeneous or on-shell massless deformation is not controlled by the same algebraic inverse. Boundary conditions, constraints, infrared regulators, or moduli data determine whether such a deformation is admissible.

4.3 Canonical field normalization

Define a canonically normalized polarization amplitude by

$$\gamma_s := \frac{q_s}{\sqrt{32\pi G_d}}. \tag{28}$$

Then

$$I^{(2)}_{\mathrm{TT}} = \frac{1}{2}\sum_s \int_k k^2 |\gamma_s(k)|^2. \tag{29}$$

The explicit G_d has disappeared from the free kinetic term and reappears in the matter coupling through $q_s = \sqrt{32\pi G_d}\,\gamma_s$. This does not invalidate the stiffness interpretation. It illustrates that a Hessian coefficient has meaning only relative to a fixed coordinate on field space. The metric perturbation q_s, rather than the canonically rescaled graviton, is fixed by the operational geometry and universal matter coupling.

More generally, if $q' = aq$, then

$$\mathcal{K}' = \frac{\mathcal{K}}{a^2}. \tag{30}$$

Any claim that reads a physical number from a field-space Hessian without fixing a is incomplete.

5 Einstein backgrounds and spectral stiffness

5.1 The physical operator

Let $\bar{g}$ be an Einstein background,

$$\bar{R}_{\mu\nu} = \frac{2\Lambda}{d-2}\bar{g}_{\mu\nu}, \tag{31}$$

with boundary conditions that make the chosen quadratic operator symmetric. The gauge-fixed metric Hessian is built from a Lichnerowicz-type operator. On the physical transverse-traceless sector it has the schematic form

$$\mathcal{K}_{\mathrm{TT}} = \frac{1}{32\pi G_d}\,\mathcal{P}_{\mathrm{TT}}\left[-\bar{\nabla}^2 + \mathcal{U}(\bar{R})\right]\mathcal{P}_{\mathrm{TT}}, \tag{32}$$

where $\mathcal{P}_{\mathrm{TT}}$ is the physical projector appropriate to the global problem and $\mathcal{U}(\bar{R})$ is algebraic in the background curvature. Its exact form depends on dimension, index convention, the background equations, and whether the quadratic functional is an action, Euclidean action, Hamiltonian, or canonical energy. None of those choices changes the principal symbol

$$\sigma_2(\mathcal{K}_{\mathrm{TT}})(x,k) = \frac{\bar{g}^{\mu\nu}k_\mu k_\nu}{32\pi G_d}\,\mathbf{1}_{\mathrm{TT}}. \tag{33}$$

Proposition 5.1 (Local curved-background scaling)**.** *For wavelengths short compared with the background curvature radius but long compared with the effective-theory cutoff, every physical spin-two polarization governed by the Einstein–Hilbert term has inverse propagator*

$$\lambda(k) = \frac{1}{32\pi G_d}\left(k^2 + O(\bar{R})\right). \tag{34}$$

The coefficient of the two-derivative principal part is $1/G_d$.

Proof. The second variation of a local two-derivative diffeomorphism-invariant metric action contains at most two derivatives of h. Normal coordinates at a point reduce its highest-derivative part to the flat Fierz–Pauli operator. Commutators of covariant derivatives and explicit cosmological terms contribute only curvature-order terms. Restriction to a physical transverse-traceless polarization yields equation (33); the WKB eigenvalue then has the stated form. □

The proposition is local. Global spectral positivity is a separate question. The spectrum depends on topology, boundary conditions, horizons, thermodynamic ensemble, and possible negative modes. For example, a Euclidean saddle can possess a physical negative mode even though the principal symbol is positive. The coefficient $1/G_d$ still multiplies the operator, but the saddle is not a stable minimum.

5.2 AdS radius and dimensionless rigidity

For an asymptotically anti-de Sitter background with curvature radius L_{AdS}, the natural dimensionless gravitational normalization is

$$\mathcal{N}_{\mathrm{grav}} := \frac{L_{\mathrm{AdS}}^{d-2}}{G_d}. \tag{35}$$

Here d is the bulk dimension. A dimensionless metric deformation varying on the AdS scale has quadratic action of order

$$I^{(2)} \sim \mathcal{N}_{\mathrm{grav}}\, h^2. \tag{36}$$

In a concrete holographic dual, $\mathcal{N}_{\mathrm{grav}}$ is related to a normalization of boundary stress-tensor correlation functions. The numerical constant depends on dimension and convention. This is a matching relation in a specified dual pair, not a universal information-theoretic formula for G_d.

5.3 Boundary modes and charges

On a manifold with boundary, a vector field that is pure gauge in the interior may act nontrivially at the boundary. The presymplectic form can then acquire a surface contribution, and the transformation can carry a Hamiltonian charge [12, 8]. Such modes are especially prominent in three-dimensional AdS gravity, where local spin-two polarizations are absent but boundary gravitons are physical. Their quadratic normalization continues to depend on the action coefficient, yet it is not captured by merely counting local transverse-traceless plane waves.

Accordingly, the physical space in equation (13) must be defined with its asymptotic symmetry group and boundary phase space. Quotienting every Lie derivative would erase charged degrees of freedom; quotienting none would leave the Hessian singular. The distinction between proper gauge and physical asymptotic symmetry is part of the stiffness data.

6 Zero modes, signs, and stability

6.1 Four notions that should not be conflated

The phrase positive gravitational Hessian can refer to several different objects:

(i) the principal symbol of a Euclidean gauge-fixed operator;

(ii) the full spectrum of that Euclidean operator with boundary conditions;

(iii) the Lorentzian Hamiltonian of physical perturbations;

(iv) the canonical energy associated with a specified background symmetry.

The first is local and is positive on transverse-traceless momenta when $G_d > 0$. The second can have global negative or zero modes. The third is defined after solving constraints and selecting physical initial data. The fourth depends on a vector field, hypersurface, gauge behavior at its boundary, and the linearized equations. Claims about one do not automatically transfer to the others.

6.2 The Euclidean conformal factor

Write a Euclidean metric locally as $g_{\mu\nu} = e^{2\phi}\hat{g}_{\mu\nu}$. The Einstein–Hilbert action contains a kinetic term for ϕ with the sign opposite to the transverse-traceless modes. Rapid conformal variations can therefore drive the naive Euclidean action downward without bound. This is the conformal-factor problem analyzed in the gravitational path integral [6]. Gauge fixing does not turn the conformal factor into a positive physical graviton polarization.

The flat kernel equation (23) was explicitly restricted to the transverse-traceless sector. It is not a theorem that the entire Euclidean metric Hessian is positive. Treatments of the gravitational path integral may rotate the conformal contour, integrate constraints, or use a reduced phase space, but each is additional structure.

6.3 Infrared and on-shell zero modes

The operator k^2/G_d vanishes at $k = 0$. There are several physically different reasons a zero mode may appear:

- a proper diffeomorphism is a redundancy;
- a Killing vector is a reducibility parameter of the gauge condition;
- a constant deformation can change a modulus or boundary datum;
- an on-shell massless wave obeys $k^2 = 0$ in Lorentzian signature;
- a family of exact solutions can provide a tangent zero mode;
- an instability threshold can produce a genuine physical zero eigenvalue.

These cases require different treatments. The first is quotiented, the second changes determinant factors, the third belongs to a finite-dimensional moduli problem, the fourth is a propagating solution rather than an invertible off-shell response, and the last two diagnose a family or a change in stability.

Remark 6.1 (Off shell versus on shell). The statement that the massless inverse propagator is k^2/G_d is an off-shell statement about the quadratic operator. Evaluating it on a Lorentzian solution gives zero because the linearized equation is satisfied. This does not say that the wave carries zero energy. Its canonical energy is quadratic in initial-data derivatives and is proportional to $1/G_d$.

6.4 Stable sectors

On a stationary background, a canonical energy can sometimes be constructed from the covariant symplectic current. Under suitable gauge, boundary, and constraint conditions it supplies a quadratic stability criterion [7]. Positive canonical energy is then a meaningful form of stiffness on the physical solution space. It is background dependent. A black hole with a negative canonical-energy perturbation or a Euclidean saddle with a negative mode is not made stable merely because $G_d > 0$.

This observation sets the correct logical direction: $1/G_d$ normalizes the quadratic form, while the differential operator, constraints, boundary data, and background determine its sign and kernel.

7 Units and normalization audit

7.1 Natural units in arbitrary dimension

The action is dimensionless in units $c = \hbar = 1$. Since $[\mathrm{d}^d x] = L^d$ and $[R] = L^{-2}$, equation (2) implies

$$[G_d] = L^{d-2} = M^{2-d}, \qquad [G_d^{-1}] = L^{2-d} = M^{d-2}. \tag{37}$$

Therefore

$$\left[\frac{k^2}{G_d}\right] = M^d, \tag{38}$$

which is exactly the dimension needed for $\int \mathrm{d}^d x\, h\mathcal{K}h$ when h is dimensionless.

For a deformation with amplitude ϵ, characteristic length L, and support volume of order L^d, two derivatives give

$$I^{(2)} \sim \frac{L^{d-2}}{G_d}\,\epsilon^2. \tag{39}$$

The dimensionless combination L^{d-2}/G_d is the integrated rigidity at scale L. It grows with the size of a coherent deformation because more spacetime volume participates, even though the local inverse propagator falls as $k^2 \sim L^{-2}$.

7.2 Restoring c and $\hbar$

In four dimensions the Lorentzian Einstein–Hilbert action is

$$\frac{S_{\mathrm{EH}}}{\hbar} = \frac{c^3}{16\pi\hbar G_4}\int \mathrm{d}^4 x\sqrt{-g}\,(R - 2\Lambda), \tag{40}$$

when the time coordinate is expressed consistently. A dimensionless deformation on a length scale L has exponent of order

$$\frac{S^{(2)}}{\hbar} \sim \frac{c^3 L^2}{\hbar G_4}\,\epsilon^2 = \frac{L^2}{\ell_{\mathrm{P}}^2}\,\epsilon^2, \qquad \ell_{\mathrm{P}}^2 := \frac{\hbar G_4}{c^3}. \tag{41}$$

Thus semiclassical metric fluctuations are small for coherent scales much larger than the Planck length, provided no soft mode or instability defeats the quadratic estimate.

7.3 Running and scheme labels

In an effective action, G_d is a renormalized coupling $G_d(\mu)$. A quotation of its value must state the scale and scheme when quantum corrections are relevant. Observables are scheme independent after all terms are combined, but the split between the Einstein coefficient, higher-curvature coefficients, and nonlocal contributions can change under renormalization and field redefinitions. The stiffness statement refers to the coefficient of the normalized two-derivative operator in that specified effective action.

8 Canonical energy and an information metric

8.1 Relative-entropy Hessian

Let $\rho(\lambda)$ be a differentiable family of faithful density operators with reference $\rho_0 = \rho(0)$. The relative entropy

$$D(\rho\|\rho_0) = \operatorname{Tr}\rho(\log\rho - \log\rho_0) \tag{42}$$

has vanishing first derivative at the reference. Its second derivative defines the Bogoliubov–Kubo–Mori quadratic form [2, 11]

$$g^{\mathrm{BKM}}_{\rho_0}(\dot\rho,\dot\rho) := \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} D(\rho(\lambda)\|\rho_0)\bigg|_{\lambda=0}. \tag{43}$$

This is a quantum information metric. It should not be silently identified with every other metric called quantum Fisher information; the symmetric logarithmic derivative and Bures metrics use different operator means in the noncommuting case.

By itself, equation (43) contains no spacetime, metric perturbation, gauge quotient, Newton coupling, or gravitational boundary condition. To compare it with a gravitational quadratic form requires a dictionary.

Definition 8.1 (Sufficient holographic dictionary data)**.** For the equality used below, dictionary data consist of:

(i) a holographic boundary theory, reference state, and boundary region;

(ii) a semiclassical code subspace containing the state path;

(iii) a differentiable reconstruction map $\dot\rho_B \mapsto h_{ab}$ into a bulk solution of the linearized equations;

(iv) the bulk action and its normalization, including G_d;

(v) asymptotic, extremal-surface, and hypersurface boundary conditions;

(vi) a gauge, such as the relevant Hollands–Wald gauge, in which the covariant-phase-space identity has no omitted corner contribution;

(vii) an order in the semiclassical expansion at which the equality is meant.

The list is intentionally concrete. Replacing it by the phrase emergence from entanglement does not define a map between the two tangent spaces.

8.2 Canonical energy

For a bulk region associated with a boundary ball B, let Σ_B be the corresponding AdS-Rindler hypersurface and ξ_B the Killing vector that generates the reference modular flow. If ω is the covariant symplectic current of the bulk action, define the canonical energy

$$\mathcal{E}_B(h_1,h_2) := \int_{\Sigma_B} \omega(\bar g; h_1, \mathcal{L}_{\xi_B} h_2). \tag{44}$$

For Einstein gravity, ω inherits the overall factor $1/(16\pi G_d)$ from the Lagrangian. Consequently

$$\mathcal{E}_B(h,h) = \frac{1}{G_d}\,\mathcal{Q}_B[h,h] \tag{45}$$

for a dimensionful geometric bilinear $\mathcal{Q}_B$ independent of the overall Newton coefficient when the background radius and normalized perturbation are held fixed.

Theorem 8.2 (Dictionary-dependent information/canonical-energy identity)**.** *Consider a holographic CFT vacuum, a ball-shaped region B, and a differentiable state path in a semiclassical code subspace. Assume the complete dictionary data of theorem 8.1, including an on-shell bulk perturbation and the gauge and boundary conditions required by the covariant-phase-space construction. At leading classical bulk order, the boundary relative-entropy Hessian satisfies*

$$g^{\mathrm{BKM}}_{\rho_{B,0}}(\dot\rho_B, \dot\rho_B) = \mathcal{E}_B(h, h). \tag{46}$$

In particular, its gravitational normalization scales as $1/G_d$.

Proof. In the stated holographic setting, boundary relative entropy equals the appropriate bulk relative entropy to the controlled semiclassical order [9]. Taking the second variation around the reference state yields the boundary BKM form. The gravitational relative-entropy variation is represented by the covariant symplectic flux, which in the specified Hollands–Wald gauge equals the canonical energy [10, 7]. The normalization follows from the overall coefficient of the Einstein Lagrangian. Every equality here uses the state-to-field map and boundary/gauge hypotheses in the statement. □

The theorem is not a universal equality between information geometry and gravity. It applies to a particular information metric, region, reference, dual pair, perturbative regime, and image of the reconstruction map. At bulk quantum order, matter and graviton relative entropy and generalized entropy enter together; isolating a pure classical metric Hessian is then no longer the entire statement.

8.3 What the equality can calibrate

Suppose a boundary stress-tensor two-point coefficient C_T is known in a fixed convention and the bulk dual is Einstein dominated. Holographic renormalization relates C_T to a constant times $L_{\mathrm{AdS}}^{d-2}/G_d$. The boundary relative-entropy Hessian for stress-tensor-generated perturbations carries the same normalization. With the duality dictionary and convention supplied, this can determine the dimensionless bulk ratio. It still does not determine a dimensionful G_d without the bulk length scale, nor does it give a formula applicable to an unrelated microscopic system.

Corollary 8.3 (Conditional pullback interpretation)**.** *Under the hypotheses of theorem 8.2, the BKM quadratic form restricted to reconstructible state directions is the pullback of bulk canonical energy. Hence $1/G_d$ is an information-geometric stiffness on that restricted tangent space. Without the reconstruction map and its normalization, the pullback is undefined.*

This corollary gives a precise version of an attractive intuition: distinguishable boundary state deformations are costly in proportion to the energy of the corresponding bulk metric perturbation. The restriction to reconstructible directions is essential. State-space directions outside the code subspace can have a perfectly well-defined information metric and no classical metric image.

9 Flat space as a nonholographic control example

Consider ordinary perturbative gravity about Minkowski space with a prescribed Newton coupling. The transverse-traceless calculation of section 4 is complete as an effective-field-theory statement:

$$I^{(2)}_{\mathrm{TT}}[q] = \frac{1}{64\pi G_d} \sum_s \int_k k^2 |q_s|^2. \tag{47}$$

No microscopic density matrix appears. One could independently choose a family of Gaussian quantum states with a BKM metric, but there is no canonical reason for that metric to equal equation (47). To assert equality one would have to specify a map from the state parameters to $q_s(k)$, including its overall scale.

Indeed, let a one-parameter microscopic coordinate x have information line element $F\,\mathrm{d}x^2$, and let a metric amplitude be reconstructed as $q = ax$. Pulling the gravitational Hessian back gives

$$\mathrm{d}s^2_{\mathrm{grav,pullback}} = a^2 \frac{k^2}{32\pi G_d}\,\mathrm{d}x^2. \tag{48}$$

For any positive F and G_d, choosing $a^2 = 32\pi G_d F/k^2$ makes the two forms equal. Conversely, changing a changes the inferred G_d. Equality therefore has empirical content only when the reconstruction normalization a is fixed independently.

Proposition 9.1 (No equality from positivity alone)**.** *The positivity of a microscopic information metric and of a physical gravitational quadratic form does not determine an equality between them, nor does it determine G_d.*

Proof. Two positive quadratic forms on different vector spaces cannot be compared without a linear map. Given a map, multiplying it by a nonzero scalar rescales the pullback quadratic form by the square of that scalar. Therefore positivity alone leaves both the map and its normalization undetermined. The one-mode example equation (48) exhibits the freedom explicitly. □

This control example is useful because it contains the same k^2/G_d stiffness as the holographic bulk but none of the data needed for equation (46). The gravitational interpretation survives; the microscopic Fisher identification does not.

10 Induced gravity and renormalized stiffness

10.1 One-loop effective action

Let a field with quadratic operator Δ_g be integrated out on a background metric. Its one-loop contribution has the schematic form

$$\Gamma_1[g] = \frac{\sigma}{2}\,\mathrm{Tr}\log\Delta_g, \tag{49}$$

where σ includes statistics and multiplicity. In a proper-time representation,

$$\mathrm{Tr}\log\Delta_g = -\int_{\epsilon^2}^{\infty}\frac{\mathrm{d}s}{s}\,\mathrm{Tr}\,e^{-s\Delta_g} + \text{constant}, \tag{50}$$

and the small-s heat kernel has an asymptotic expansion

$$\mathrm{Tr}\,e^{-s\Delta_g} \sim \frac{1}{(4\pi s)^{d/2}}\int \mathrm{d}^d x\sqrt{g}\,\left(a_0 + a_1 s + a_2 s^2 + \cdots\right). \tag{51}$$

For Laplace-type operators, a_1 contains a curvature term. It therefore renormalizes the coefficient of $\int\sqrt{g}R$, and hence inverse Newton's constant. The sign and magnitude depend on spin, nonminimal curvature couplings, and the regularization and subtraction prescription.

The renormalized coefficient may be organized as

$$\frac{1}{G_d(\mu)} = \frac{1}{G_{d,\mathrm{bare}}} + \frac{1}{G_{d,\mathrm{ind}}}(\mu;\{m_i,s_i,\xi_i\},\mathcal{R}), \tag{52}$$

where m_i, s_i, and ξ_i represent masses, spins, and curvature couplings, and $\mathcal{R}$ denotes regulator and renormalization data. The split between bare and induced pieces is not itself observable. Only the renormalized effective action, matched to measurements or a UV theory, is.

10.2 Sakharov's elasticity analogy

Sakharov proposed that gravitational dynamics could arise as an elasticity of the vacuum induced by quantum fluctuations [13]. Later work developed spectral and stress-tensor representations of the induced Einstein term [1, 15, 14]. In the language of this paper, these calculations generate or renormalize the coefficient multiplying the two-derivative metric Hessian. They therefore provide an explicit mechanism for metric stiffness.

The analogy should not be asked to do more than the calculation. If a cutoff regularization gives a contribution of order

$$\frac{1}{G_{d,\mathrm{ind}}} \sim N\Lambda_{\mathrm{UV}}^{d-2}, \tag{53}$$

the numerical coefficient depends on species and couplings, and the power divergence can be shifted by a local counterterm. A physical prediction needs a UV completion or a renormalization condition that fixes the total coefficient. Counting species without this information does not determine Newton's constant.

10.3 Thresholds, nonminimal couplings, and nonlocality

Massive fields contribute differently above and below their thresholds. Nonminimally coupled scalars with operator $-\nabla^2 + m^2 + \xi R$ change the curvature heat-kernel coefficient as ξ changes. Gauge fields and gravitons require their gauge and ghost sectors. Massless fields can also produce nonlocal terms such as

$$\int \sqrt{g}\, R \log(-\nabla^2/\mu^2) R, \tag{54}$$

whose quadratic kernel is not captured by a constant shift of $1/G_d$. These facts are not defects of induced gravity; they are the ordinary content of an effective action.

Proposition 10.1 (Additivity and ambiguity of induced stiffness)**.** *At a fixed perturbative order and in a fixed scheme, independent field determinants contribute additively to the coefficient of the local Einstein term. Nevertheless, the decomposition* $G_d^{-1} = G_{d,\mathrm{bare}}^{-1} + G_{d,\mathrm{ind}}^{-1}$ *does not determine either summand from the measured total.*

Proof. For independent quadratic fields the partition function factorizes and the effective action is a sum of trace logarithms. Heat-kernel coefficients, and hence the induced local R coefficients, add. But for any finite shift δ, replacing $G_{d,\mathrm{bare}}^{-1}$ by $G_{d,\mathrm{bare}}^{-1} + \delta$ and $G_{d,\mathrm{ind}}^{-1}$ by $G_{d,\mathrm{ind}}^{-1} - \delta$ leaves the total unchanged. A renormalization condition or UV matching prescription is needed to select a split. □

11 Higher derivatives and the domain of Einstein stiffness

11.1 Local corrections

A gravitational effective action contains operators such as

$$\Gamma_{\mathrm{hd}} = \int \mathrm{d}^d x \sqrt{-g}\, \left[\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \cdots\right]. \tag{55}$$

This derivative expansion is the standard effective-field-theory treatment of general relativity [4]. Around flat space their quadratic contributions scale schematically as

$$\mathcal{K}_{\mathrm{TT}}(k) = \frac{k^2}{32\pi G_d} + \alpha_{\mathrm{TT}} k^4 + O(k^6/\Lambda_{\mathrm{EFT}}^{d+2}), \tag{56}$$

with coefficient dimensions understood. At momenta satisfying

$$|\alpha_{\mathrm{TT}}|k^2 \ll \frac{1}{32\pi G_d}, \tag{57}$$

the Einstein term controls the stiffness. Outside that window the full kernel, not k^2/G_d alone, determines the response.

One should not necessarily interpret extra roots of a truncated polynomial kernel as new fundamental particles. In an effective theory, higher derivatives are used perturbatively below the cutoff; resumming a finite truncation can create spurious poles beyond its domain of validity. Conversely, if a genuine light scalar or spin-two field is present, it should be retained explicitly and the quadratic form becomes a coupled block matrix.

11.2 Mixed metric–matter Hessians

Suppose a scalar background has a nonzero gradient or a nonminimal coupling. Then the quadratic action may be

$$\frac{1}{2}\begin{pmatrix} h & \delta\phi \end{pmatrix}\begin{pmatrix} \mathcal{K}_{hh} & \mathcal{K}_{h\phi} \\ \mathcal{K}_{\phi h} & \mathcal{K}_{\phi\phi} \end{pmatrix}\begin{pmatrix} h \\ \delta\phi \end{pmatrix}. \tag{58}$$

Integrating out $\delta\phi$ at quadratic order gives the Schur complement

$$\mathcal{K}_{\mathrm{eff},h} = \mathcal{K}_{hh} - \mathcal{K}_{h\phi}\mathcal{K}_{\phi\phi}^{-1}\mathcal{K}_{\phi h}, \tag{59}$$

when the inverse exists. The metric susceptibility can then differ substantially from G_d/k^2, especially near a soft matter mode. The Einstein coefficient remains the two-derivative normalization within $\mathcal{K}_{hh}$, but it is not the only contribution to the observable response.

11.3 Field redefinitions

Local field redefinitions can move terms between higher-curvature operators and matter couplings. On-shell amplitudes remain invariant, while the off-shell Hessian in a chosen variable changes. This is another reason to tie the stiffness statement to the operational metric that appears in the line element and to specify the effective-action basis. At leading two-derivative order with minimal universal matter coupling, the Einstein coefficient has a clear meaning; beyond that order the basis and matching convention are part of the claim.

Hypothesis 11.1 (Einstein-dominated infrared regime)**.** There exists a momentum window in which the reconstructed metric is the only relevant spin-two field, the physical quadratic kernel is nonsingular away from expected massless modes, and higher-derivative and nonlocal corrections are parametrically smaller than the Einstein term.

This is an effective-theory hypothesis, not a consequence of information geometry. When it holds, $1/G_d$ supplies the leading stiffness. When it fails, a single Newton coefficient is an inadequate summary of the response.

12 Why no numerical value of Newton's constant is derived

There are three conceptually distinct steps:

(1) identify the coefficient multiplying the physical metric Hessian;

(2) derive or match that coefficient from microscopic data;

(3) express the matched coefficient as a dimensionful number in a fixed unit convention and at a specified renormalization scale.

The stiffness calculation in section 4 accomplishes the first. The holographic theorem accomplishes part of the second only for a specified dual pair and dictionary. An induced-gravity calculation can also contribute to the second when its spectrum, UV prescription, bare term, and matching data are supplied. None of the general arguments performs the third.

Theorem 12.1 (Underdetermination of G_d from abstract stiffness)**.** *Assume only that a reconstructed infrared metric has a positive physical two-derivative quadratic form and that a microscopic state family has a positive information metric. Then neither the dimensionful value of G_d nor a dimensionless ratio L^{d-2}/G_d is determined.*

Proof. The two quadratic forms live on different tangent spaces. A reconstruction differential $D\mathfrak{R}$ is required to pull the gravitational form back. If $D\mathfrak{R}$ is admissible, then $aD\mathfrak{R}$ for any nonzero constant a changes the pulled-back metric by a^2. The inferred value of G_d can therefore be changed continuously while preserving positivity. Even if a dictionary fixes $D\mathfrak{R}$ and determines a dimensionless coefficient, a dimensionful value requires a physical length or energy scale. Finally, radiative corrections permit shifts between a bare Einstein coefficient and induced local counterterms while preserving the same renormalized total. The stated assumptions fix none of these choices. □

Corollary 12.2 (Data needed for a numerical determination)**.** *A numerical determination requires, at minimum, a normalized microscopic model or measured correlator, a state-to-metric dictionary, a physical length or energy standard, the relevant spectrum and couplings, a regulator or UV completion, a renormalization prescription and scale, and a matching condition for the total Einstein coefficient.*

The theorem does not say that Newton's constant can never be derived in a complete theory. It says that the general stiffness and information-geometric arguments do not constitute such a derivation. A UV-complete duality could in principle provide all of the missing data. A laboratory measurement supplies a matching condition directly. What is excluded is a number obtained from positivity, state-space curvature, a species count, or dimensional analysis without those inputs.

13 Failure modes and counterexamples

This section collects tests that a stiffness claim should survive.

13.1 Unquotiented diffeomorphisms

Take $h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)}$ of compact support in flat space. It is a proper gauge perturbation and has zero quadratic action on the background equations. The full Hessian therefore has zero eigenvalues no matter how small G_d is. Calling its determinant the stiffness before gauge fixing is meaningless.

13.2 The homogeneous mode

For $k = 0$, equation (23) vanishes. A constant metric change can represent a coordinate redundancy, a change in compactification modulus, or a change of boundary geometry. The derivative coefficient $1/G_d$ does not by itself supply a restoring force. Cosmological, curvature, boundary, or matter terms determine the homogeneous sector.

13.3 The conformal direction

A pure trace Euclidean perturbation is not one of the polarizations used in theorem 4.1. Its wrong-sign kinetic term is a direct counterexample to the claim that the unreduced Einstein action is a positive information metric. The physical transverse-traceless statement remains valid.

13.4 Low-dimensional gravity

In $d = 3$, equation (17) gives no local graviton. Nevertheless the Einstein action has a coefficient $1/G_3$ and AdS boundary charges can depend on L/G_3. This separates local plane-wave stiffness from global and boundary gravitational normalization. In $d = 2$, an Einstein term alone cannot furnish local metric dynamics.

13.5 A freely rescaled dictionary

Suppose a microscopic tangent x is mapped to $q = ax$. Without an independent normalization of a, equation (48) permits any inferred G_d. This defeats a generic Fisher-to-Newton identification even when both quadratic forms are positive and finite.

13.6 Higher-derivative dominance

If $|\alpha_{\mathrm{TT}}|k^2 \gg 1/(32\pi G_d)$, the response is controlled by $\alpha_{\mathrm{TT}}k^4$, not the Einstein term. Calling $1/G_d$ the full stiffness at that momentum omits the dominant operator. The terminology is appropriate only in the infrared window equation (57).

13.7 A soft mixed field

If $\mathcal{K}_{\phi\phi}$ has a small eigenvalue, the Schur complement equation (59) can be large or nonlocal. The metric response can then be dominated by mixing even at small momentum. The claim must either retain the coupled system or explain why such light fields are absent.

13.8 An unspecified induced sector

Two microscopic spectra with different spins and nonminimal couplings can produce different heat-kernel a_1 coefficients while having the same number of fields. Conversely, a bare counterterm can make their renormalized Newton constants equal. Species count alone is therefore not a numerical prediction.

14 Synthesis

The complete logical chain can be stated compactly. Begin with a normalized infrared metric and a specified effective action. Expand about a solution so that the linear term vanishes. Remove proper diffeomorphism directions and separate constraints from physical modes. In the Einstein-dominated window, the physical spin-two principal symbol is $k^2/(32\pi G_d)$ in the explicit flat-space normalization used here. Its inverse gives the familiar G_d/k^2 susceptibility to stress. Curvature, boundaries, zero modes, and additional fields determine the global spectrum but do not change the Einstein coefficient of the principal symbol.

A microscopic information metric enters only through a map. In the holographic ball construction, the duality, code subspace, reconstruction map, linearized equations, gauge, and boundary terms provide that map, and the BKM relative-entropy Hessian equals bulk canonical energy. That equality makes the $1/G_d$ normalization information-geometric for the reconstructible directions in the specified dual pair. A generic positive state-space metric does not provide the same identification.

Induced gravity supplies a complementary mechanism. Quantum determinants generate and renormalize the curvature term, so vacuum fluctuations contribute to metric stiffness. Their contribution is model and scheme dependent, and the bare-plus-induced split is not observable. A complete UV theory or a renormalization condition is required to determine the total.

These distinctions support the phrase in the title without inflating it. Newton's constant is the stiffness of the infrared metric in the same precise sense that a gradient coefficient is the stiffness of a massless field: it normalizes the physical quadratic inverse propagator. It is not the eigenvalue of the unreduced Hessian, not a guarantee of stability on every background, not automatically equal to an arbitrary quantum Fisher metric, and not numerically predicted by the interpretation.

15 Conclusion

The Einstein–Hilbert coefficient has a direct quadratic meaning. With $g = \bar{g} + h$, normalized transverse-traceless polarizations, and the Hessian convention $I^{(2)} = \frac{1}{2}h\mathcal{K}h$, flat Euclidean space gives

$$\mathcal{K}_s(k) = \frac{k^2}{32\pi G_d}. \tag{60}$$

The corresponding source response has magnitude $16\pi G_d T_s^{\mathrm{TT}}/k^2$. On curved Einstein backgrounds the Lichnerowicz-type operator retains the same $1/G_d$ normalization and two-derivative principal symbol, while curvature and boundary data control its lower-order spectrum.

The qualifications are part of the result. Proper diffeomorphisms must be removed; asymptotic symmetries with charges must not be removed indiscriminately. The Euclidean conformal factor is not a positive physical polarization. Massless and moduli zero modes need infrared or boundary data. Higher derivatives and mixed light fields can dominate outside the Einstein window. Metric normalization must be fixed before its Hessian coefficient is assigned physical meaning.

The information-geometric comparison is sharper than an analogy only in a specified dictionary. For the holographic vacuum ball and a controlled semiclassical code subspace, the BKM relative-entropy Hessian equals bulk canonical energy, whose symplectic normalization is proportional to $1/G_d$. This is a theory-dependent equivalence with explicit gauge, boundary, and reconstruction assumptions. It does not extend automatically to a nonholographic state family.

Finally, induced gravity shows how quantum matter contributes to the same coefficient but also exposes why a number does not follow. Spectrum, couplings, cutoff or UV completion, bare counterterms, thresholds, and a renormalization condition all matter. The defensible conclusion is therefore structural rather than numerical: in a normalized, gauge-reduced, Einstein-dominated infrared description, $1/G_d$ measures the resistance of the physical metric to deformation.

A Flat-space projector and factor audit

This appendix checks the coefficient in equation (23). Begin with equation (2) at $\Lambda = 0$ and write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The expansions

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu}{}_{\rho}h^{\rho\nu} + O(h^3), \tag{61}$$

$$\sqrt{-g} = 1 + \frac{1}{2}h + \frac{1}{8}(h^2 - 2h_{\mu\nu}h^{\mu\nu}) + O(h^3) \tag{62}$$

inserted into the curvature scalar yield equation (18) after integrating by parts. For a transverse-traceless spatial perturbation, $h = 0$, $\partial_i h_{ij} = 0$, and $h_{0\mu} = 0$. The last three terms of equation (18) vanish and the first becomes

$$S^{(2)}_{\mathrm{TT}} = -\frac{1}{64\pi G_d}\int \mathrm{d}^d x\, \partial_\lambda h^{\mathrm{TT}}_{ij}\partial^\lambda h^{\mathrm{TT}}_{ij}, \tag{63}$$

which is equation (19) in signature $(-+\cdots+)$.

In Euclidean signature use

$$h^{\mathrm{TT}}_{ij}(x) = \sum_s \int \frac{\mathrm{d}^d k}{(2\pi)^d} q_s(k) e^s_{ij}(k) e^{ik\cdot x}, \tag{64}$$

with $q_s(-k) = q_s(k)^*$ for a real field. Orthogonality gives

$$\int \mathrm{d}^d x\, \partial_\lambda h^{\mathrm{TT}}_{ij}\partial_\lambda h^{\mathrm{TT}}_{ij} = \sum_s \int \frac{\mathrm{d}^d k}{(2\pi)^d} k^2 |q_s(k)|^2. \tag{65}$$

Comparing

$$\frac{1}{64\pi G_d} k^2 |q_s|^2 = \frac{1}{2} q_s^* \left(\frac{k^2}{32\pi G_d}\right) q_s \tag{66}$$

proves the factor. If instead the polarization convention is $e^s_{ij} e^{s'ij} = 2\delta^{ss'}$, the mode amplitude and displayed factor change accordingly. The tensor action and all observables are unchanged.

For nonzero spatial momentum in $n = d - 1$ dimensions, define

$$\pi_{ij} = \delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}. \tag{67}$$

The spatial transverse-traceless projector is

$$P^{\mathrm{TT}}_{ij,kl} = \frac{1}{2}(\pi_{ik}\pi_{jl} + \pi_{il}\pi_{jk}) - \frac{1}{n-1}\pi_{ij}\pi_{kl}. \tag{68}$$

It is symmetric and idempotent, annihilates k^i, and has zero trace in either index pair. It is undefined at $\boldsymbol{k} = 0$, another explicit sign that the homogeneous sector requires separate data.

B Gaussian response and Fisher pullbacks

Consider a finite-dimensional Euclidean Gaussian model

$$Z[J] = \int \mathrm{d}^n q\, \exp\left[-\frac{1}{2} q^T K q + J^T q\right], \tag{69}$$

where K is symmetric positive definite. Completing the square gives

$$\langle q \rangle_J = K^{-1} J, \qquad \langle (q - \langle q \rangle)(q - \langle q \rangle)^T \rangle = K^{-1}. \tag{70}$$

Thus the inverse propagator, response stiffness, and inverse covariance are the same matrix. For a Gaussian location family with fixed covariance $C = K^{-1}$, the classical Fisher metric of the mean parameter is K. This is the clean model behind much information-geometric intuition.

But a gravitational application still needs a map. If microscopic parameters x^a reconstruct Gaussian means $q^A = R^A{}_a x^a$, the pulled-back metric is

$$F_{ab}^{\text{pullback}} = R^A{}_a K_{AB} R^B{}_b. \tag{71}$$

Changing R changes the metric. If R has a kernel, microscopic directions in that kernel have no metric image. If R is not surjective, some gravitational perturbations are not reconstructible. If gauge directions occur in the range, the quotient must be taken before an inverse is defined. The holographic dictionary in theorem 8.1 supplies a highly structured counterpart of R; abstract information geometry does not.

For the one-mode gravity kernel, take

$$K = \frac{k^2}{32\pi G_d}. \tag{72}$$

The covariance is $32\pi G_d/k^2$, while the stress-tensor convention in equation (10) inserts an additional factor of one half and produces the physical response $16\pi G_d T/k^2$. This accounts for the factor difference between inverse covariance and response to the conventionally normalized stress tensor.

C Executable arithmetic model

The accompanying Haskell modules implement the algebraic boundary of the claims using exact rational arithmetic. They do not simulate tensor gravity, solve differential equations, or certify a holographic duality. Their purpose is to make normalizations and logical gates auditable.

The core module evaluates

$$K(k^2) = \frac{k^2}{32\pi_* G}, \qquad I(q) = \frac{k^2 q^2}{64\pi_* G}, \qquad q(T) = \frac{16\pi_* G T}{k^2}, \tag{73}$$

where π_* is supplied as an exact positive rational surrogate. Keeping π_* symbolic in this way permits exact equality tests without pretending that π is rational. The code verifies

$$I(q) = \frac{1}{2} K q^2, \qquad K q(T) = \frac{1}{2} T, \tag{74}$$

where the second identity includes the half-normalized stress coupling. It rejects $G \leq 0$, $\pi_* \leq 0$, negative Euclidean k^2, and inversion at $k^2 = 0$.

A dictionary record has separate flags for a code subspace, a bulk map, a canonical-energy identity, and metric normalization. The program returns an information/canonical-energy match only when every item is present. This is a logical model of the scope of theorem 8.2, not evidence for its physical assumptions. The induced-gravity function adds a bare inverse coupling and a list of loop contributions; a deterministic test constructs two different decompositions with the same total to demonstrate theorem 10.1.

The property suite checks linearity in k^2, inverse scaling with G, the Hessian/action identity, source inversion, length scaling, field-rescaling covariance, additivity of induced contributions, the exact k^4 correction, and refusal under an incomplete dictionary. QuickCheck uses a recorded seed and bounded positive integers. These tests establish arithmetic consistency of the finite model only. The analytic projector, gauge quotient, curved background, and holographic hypotheses remain mathematical and physical arguments in the paper.

Statement	Required domain	Excluded inference
$\mathcal{K}_s = k^2/(32\pi G_d)$	Flat background, normalized TT mode, nonzero Euclidean momentum	Positivity of the unreduced metric Hessian
Curved principal symbol	Einstein background, physical projector, EFT window	Global spectral stability
$q \sim G_d T/k^2$	Fixed source convention and invertible nonzero mode	Algebraic response of a homogeneous or on-shell mode
$g^{\mathrm{BKM}} = \mathcal{E}_B$	Holographic vacuum ball, code subspace, on-shell map, gauge and boundary data	Generic information metric equals gravity
Induced $1/G_d$	Specified spectrum, couplings, regulator and matching	Universal species-count prediction
$1/G_d$ as stiffness	Normalized physical metric, Einstein-dominated infrared action	Numerical value of Newton's constant

D Scope ledger

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