Renormalization of Reconstructed Geometry and Metric Beta Functionals

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Abstract

A reconstruction scale can enter an infrared description in several inequivalent ways. Wilson coefficients may run while the metric argument of the action is held fixed. A renormalized functional may satisfy a Callan–Symanzik equation. Local sources may obey a local renormalization-group equation with a Weyl anomaly. Separately, a scale-indexed reconstruction may assign different metrics to the same microscopic state. This paper distinguishes these four statements and asks when the last one can be expressed by a metric beta functional whose fixed points coincide with the infrared metric equation. The answer is conditional. Metrics reconstructed at different scales must be compared by a specified connection modulo diffeomorphisms and other redundant field redefinitions. The resulting beta functional must be a local covariant vector on the physical reconstruction space. Finally, it must be related to the variational reconstruction defect by a nondegenerate local bundle map. Under these hypotheses, vanishing of the physical metric beta functional is equivalent to vanishing of the defect. Positivity of the bundle map gives a monotonicity statement only in a gauge-fixed Euclidean or otherwise positive field-space setting; it is not automatic for Lorentzian metrics. We prove field-redefinition covariance, analyze redundant operators and scheme changes, give fixed-point examples, and construct counterexamples when the gradient relation is absent or degenerate. The sigma-model and string-background beta functions of Friedan and Callan–Friedan–Martinec–Perry provide the precise analogy: target-space backgrounds are world-sheet couplings, and Weyl anomaly coefficients reproduce coupled background equations in an appropriate scheme. They do not establish a universal identity between arbitrary coarse-graining flows and the Einstein equation.

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1 Introduction

The word "renormalization" is used for several operations that are related but not identical. A Wilsonian action changes when modes are integrated out. A renormalized correlation function satisfies a differential equation expressing independence of an arbitrary subtraction scale. A local renormalization group promotes couplings to spacetime-dependent sources and studies local Weyl transformations. A geometric reconstruction can also depend on its resolution: the same microscopic state may be assigned different continuum metrics at different coarse-graining scales. Confusing these operations turns a useful analogy into the unsupported formula

$$\beta^{(g)}_{\mu\nu} \stackrel{?}{=} G_{\mu\nu} + \Lambda g_{\mu\nu} - 8\pi G_{\mathrm{N}} T_{\mu\nu}. \tag{1}$$

There is no general reason for equation (1) to hold. Related programs have connected holographic radial evolution to Wilsonian renormalization and real-space entanglement renormalization to emergent bulk geometry [8, 9]. Those domain-specific constructions motivate the question addressed here; they do not, by themselves, assert or imply the universal identity equation (1).

The modern Wilsonian account organizes theories by flows on a space of actions or couplings [1, 4]. The Callan–Symanzik equation instead describes the response of renormalized quantities to a subtraction scale [2, 3]. Redundant operators and field redefinitions mean that coordinates on theory space are not physical observables [5, 6]. These distinctions remain necessary in gravitational effective field theory, where local redefinitions of the metric move terms among curvature operators while leaving on-shell observables unchanged.

There is nevertheless a genuine and important metric beta function in the nonlinear sigma model. The target metric is an infinite collection of world-sheet couplings, and its leading beta function is proportional to its Ricci tensor [10, 11]. For string backgrounds, vanishing of the relevant Weyl anomaly coefficients gives coupled equations for the target metric, antisymmetric tensor, dilaton, and any other retained backgrounds [12, 13, 15]. The relation to a target-space action is itself structured and scheme sensitive. Raw beta functions, Weyl anomaly coefficients, and Euler–Lagrange derivatives are not interchangeable term by term.

The purpose of this paper is to state exactly what must be added before an analogous statement is valid for reconstructed geometry. The metric equation derived from an infrared effective action is packaged as a reconstruction defect

$$\mathcal{C}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} + H^{\mathrm{loc}}_{\mu\nu} + H^{\mathrm{nl}}_{\mu\nu} - 8\pi G_{\mathrm{N}} T_{\mu\nu}. \tag{2}$$

This definition carries the inverse-metric variation and stress-tensor conventions reviewed in section 4. The central question is not whether one can name some scale derivative β_g. It is whether a scale derivative is well defined on the quotient by diffeomorphisms and redundant field directions, and whether it is related to the variational covector represented by $\mathcal{C}_{\mu\nu}$.

The main theorem assumes a local reconstruction RG generator and a relation

$$[\widehat{\beta}_g] = -\mathcal{M}_g[\mathbf{C}^{\mathrm{RG}}_g] \quad \text{on the physical field space,} \tag{3}$$

where $\mathbf{C}^{\mathrm{RG}}_g$ is the local Wilsonian defect covector, brackets denote quotient classes, and $\mathcal{M}_g$ is injective on the admissible physical cotangent space. Under these assumptions, $[\widehat{\beta}_g] = 0$ if and only if $[\mathbf{C}^{\mathrm{RG}}_g] = 0$. Equation equation (3) is an additional gradient or variational hypothesis, not a consequence of coarse-graining, information geometry, or the existence of a reconstruction map.

Three qualifications are central. First, a fixed point is defined only modulo redundant transformations. A beta functional equal to a Lie derivative may be physically zero. Second, nondegeneracy is essential: a singular mobility can annihilate a nonzero defect. Third, fixed-point equivalence is weaker than a monotonic gradient theorem. Monotonicity additionally requires a symmetric positive field-space pairing and control of explicit scale dependence and all other running sources. The natural DeWitt pairing for Lorentzian metrics is not positive on the unreduced field space.

The paper is organized as follows. Sections 2–3 define the scale-indexed reconstruction and separate four notions of running. Section 4 fixes the variational defect and its Ward property. Sections 5 and 6 treat field redefinitions, redundant operators, and scheme changes. Section 7 defines the local RG structure used in the theorem. Section 8 states and proves the conditional fixed-point result. Counterexamples and examples appear in Sections 9 and 10. Sections 11 and 12 compare the construction with sigma models and string backgrounds. The remaining sections discuss nonlocality, boundaries, limitations, and the relation to the preceding reconstruction results.

2 A scale-indexed reconstruction bundle

Let q denote microscopic data, including a net of observables, a state, and any coarse-graining prescription needed to define an infrared description. At positive scale μ, a reconstruction is a map

$$\mathfrak{R}_\mu : q \longmapsto \big(M, g_\mu, \Phi_\mu; c^I(\mu)\big). \tag{4}$$

The symbol μ may denote an energy resolution, an inverse length, or a subtraction scale only after the construction specifies which one. We take $t = \log(\mu/\mu_0)$ as a dimensionless flow parameter.

The values g_μ and $g_{\mu'}$ lie in different fibers unless the reconstruction supplies a way to compare them. Even if the underlying manifold M is fixed, representatives can differ by a scale-dependent diffeomorphism. Matter fields can mix, coordinates on field space can change, and the set of retained variables can jump at thresholds. A derivative such as $\mu\partial_\mu g_{\mu\nu}$ is therefore not intrinsically defined by equation (4) alone.

Definition 2.1 (Reconstruction comparison structure)**.** A comparison structure on a scale interval I consists of:

(i) a fixed differentiable manifold M or specified identification maps $\mathcal{I}_{\mu\to\mu'} : M_\mu \to M_{\mu'}$;

(ii) a bundle $\pi : \mathcal{F} \to I$ whose fiber contains the reconstructed metric, retained fields, and local couplings;

(iii) a connection $\mathcal{D}_t$ on this bundle;

(iv) a redundant distribution $\mathcal{V} \subset T\mathcal{F}$ generated by diffeomorphisms and invertible local field redefinitions that do not change the physical description.

The connection records what is held fixed while the scale changes. It can subtract a compensating diffeomorphism, a wave-function normalization, or a chosen change of operator basis. The physical tangent space is the quotient

$$T_{[\Psi]}\mathcal{Q} := T_\Psi\mathcal{F}/\mathcal{V}_\Psi, \qquad \Psi = (g, \Phi, c^I). \tag{5}$$

The quotient may require gauge fixing and boundary conditions to be a genuine manifold. We use it formally only on a regular stratum where the stabilizer type is constant.

Definition 2.2 (Metric reconstruction beta functional)**.** Given theorem 2.1, the contravariant metric beta functional is

$$\widehat{\beta}_g^{\mu\nu}(x) := \mathcal{D}_t g^{\mu\nu}(x). \tag{6}$$

Its physical content is the quotient class $[\widehat{\beta}_g] \in T_{[g]}\mathcal{Q}$. It is local to derivative order N if it is a covariant expression in the retained fields and their derivatives through order N, with coefficients depending on the local couplings.

We use the inverse metric in equation (6) to match the variational convention $\delta/\delta g^{\mu\nu}$. For the covariant metric,

$$\mathcal{D}_t g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\widehat{\beta}_g^{\rho\sigma}. \tag{7}$$

Failing to track this minus sign can reverse a purported gradient formula.

Remark 2.3 (No derivative without comparison)*.* The scale-indexed family equation (4) does not determine equation (6). Two connections differing by a redundant vector field give different representatives and the same physical quotient class. Two connections differing by a nonredundant horizontal vector define genuinely different flows.

Proposition 2.4 (Pure gauge running)**.** *If $g_t = \varphi_t^* g_0$ for a family of diffeomorphisms generated by ξ_t, then*

$$\partial_t g_t = \mathcal{L}_{\xi_t} g_t \tag{8}$$

and the physical metric beta functional vanishes in the quotient by diffeomorphisms.

Proof. The derivative of a pullback along a diffeomorphism flow is the Lie derivative. By definition, Lie derivatives lie in the redundant distribution. Thus $[\partial_t g_t] = 0$ even though the displayed tensor need not vanish pointwise. $\square$

This elementary observation already rules out identifying a raw component beta with a covariant equation of motion. The fixed-point condition must be stated on the physical quotient.

3 Four distinct forms of scale dependence

3.1 Wilsonian action flow

Let S_Λ be a regulated microscopic action. Splitting fields into modes above and below an infrared cutoff k, a Wilsonian action is defined by

$$e^{-S_k[\varphi_<]} = \int \mathcal{D}\varphi_> \, e^{-S_\Lambda[\varphi_< + \varphi_>]}. \tag{9}$$

Changing k changes a functional on theory space,

$$k\frac{\mathrm{d}}{\mathrm{d}k} S_k = \mathcal{B}_k[S_k], \tag{10}$$

where the form of $\mathcal{B}_k$ depends on the blocking or cutoff prescription. Expanding in an operator basis,

$$S_k = \sum_I c^I(k)\mathcal{O}_I, \qquad k\frac{\mathrm{d}c^I}{\mathrm{d}k} = \beta^I_{\mathrm{W}}(c), \tag{11}$$

is a flow of Wilson coefficients. If the action is a functional of a metric argument, one may keep that metric argument fixed while all $c^I(k)$ run. Therefore

$$\beta^I_{\mathrm{W}} \neq 0 \quad \not\Rightarrow \quad \widehat{\beta}^{\mu\nu}_g \neq 0. \tag{12}$$

Gravity supplies a simple example. Curvature-squared coefficients depend on scale and scheme, while $g_{\mu\nu}$ can remain the field coordinate at which the Wilsonian action is evaluated. Running of c_{R^2} is not motion of the spacetime geometry.

3.2 Callan–Symanzik scale dependence

For a renormalized one-particle-irreducible functional $\Gamma_R[\phi; c, \mu]$, a schematic Callan–Symanzik equation is

$$\left[\mu\frac{\partial}{\partial\mu} + \beta^I(c)\frac{\partial}{\partial c^I} - \int \mathrm{d}^d x\, \gamma^A{}_B(c)\phi^B(x)\frac{\delta}{\delta\phi^A(x)}\right]\Gamma_R = \mathcal{A}_{\mathrm{CS}}. \tag{13}$$

The anomaly term vanishes in settings where there is no relevant explicit breaking. Equation equation (13) expresses independence of bare physics from the arbitrary scale used to parametrize renormalized quantities. It is not the Euler–Lagrange equation $\delta\Gamma_R/\delta\phi = 0$.

In particular, evaluating equation (13) on an arbitrary off-shell metric does not make that metric solve the gravitational field equation. The Callan–Symanzik operator differentiates the functional and its parameters; the metric Euler derivative differentiates with respect to a field argument.

3.3 Local renormalization group

Promote couplings c^I to local sources $c^I(x)$ and let $\bar{g}_{\mu\nu}$ be a background source metric. The local Weyl operator contains

$$\Delta^{\mathrm{W}}_\sigma = \int \mathrm{d}^d x\sqrt{-\bar{g}}\, 2\sigma\bar{g}_{\mu\nu}\frac{\delta}{\delta\bar{g}_{\mu\nu}}, \tag{14}$$

while the beta part has the schematic form

$$\Delta^\beta_\sigma = \int \mathrm{d}^d x\sqrt{-\bar{g}}\left[\sigma\beta^I(c)\frac{\delta}{\delta c^I} + (\partial_\mu\sigma)\mathcal{J}^\mu + \cdots\right]. \tag{15}$$

The local RG equation is

$$\left(\Delta^{\mathrm{W}}_\sigma - \Delta^\beta_\sigma\right)W = \mathcal{A}_\sigma. \tag{16}$$

Because Weyl transformations commute, the anomaly and beta data obey Wess–Zumino consistency conditions. Osborn's construction demonstrates how these conditions constrain local RG data [7]. The Weyl variation of the background source metric in equation (14) is not by itself an RG flow of a reconstructed dynamical metric.

3.4 Flow of reconstructed geometry

An actual reconstruction flow holds fixed specified microscopic data q and compares the outputs of $\mathfrak{R}_\mu(q)$. With the comparison structure of theorem 2.1, it is

$$\widehat{\beta}_g(q;\mu) = \mathcal{D}_t\left[\pi_g \mathfrak{R}_\mu(q)\right]. \tag{17}$$

Nothing in Wilsonian flow, the Callan–Symanzik equation, or the local Weyl identity constructs the map $\mathfrak{R}_\mu$ or its connection. A microscopic model may relate them, but that relation is additional data.

Proposition 3.1 (Logical independence)**.** *Subject to ordinary regularity assumptions, each of the following can occur:*

(a) running Wilson coefficients with a fixed reconstructed metric;

(b) a nontrivial reconstructed metric flow with fixed dimensionless local couplings;

(c) a Callan–Symanzik equation on off-shell backgrounds with a nonzero metric defect;

(d) a local Weyl anomaly with no chosen reconstruction map.

Proof. For (a), choose a fixed background field argument and a Wilsonian action with a running curvature-squared coefficient. For (b), apply a scale-dependent nonredundant smoothing map to a family of metrics while holding the displayed couplings fixed. For (c), the functional identity equation (13) holds before the metric equation is imposed. For (d), local RG is defined for ordinary QFT generating functionals on background sources, whether or not those sources arise from a reconstruction. These examples use distinct definitions and establish no implication among them. □

4 The variational reconstruction defect

We now fix the metric equation to which a beta functional might be related. Use signature $(-,+,\ldots,+)$ and vary the inverse metric. Then

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}. \tag{18}$$

Let the renormalized infrared functional be

$$\Gamma_{\mathrm{IR}}[g,\Phi;c] = \Gamma_{\mathrm{g}}[g;c] + \Gamma_{\mathrm{m}}[g,\Phi;c] + \Gamma_{\mathrm{nl}}[g,\Phi;c]. \tag{19}$$

The stress tensor is

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta\Gamma_{\mathrm{m}}}{\delta g^{\mu\nu}}. \tag{20}$$

Equivalently, for the metric variation at fixed retained matter fields and up to boundary terms, this inverse-metric convention reads $\delta\Gamma_{\mathrm{m}} = -(1/2)\int \mathrm{d}^d x\sqrt{-g}\, T_{\mu\nu}\delta g^{\mu\nu}$. Normalize the gravitational and nonlocal metric variations so that

$$\mathcal{E}_{\mu\nu} := \frac{16\pi G_{\mathrm{N}}}{\sqrt{-g}}\frac{\delta(\Gamma_{\mathrm{g}}+\Gamma_{\mathrm{nl}})}{\delta g^{\mu\nu}}. \tag{21}$$

The reconstruction defect is

$$\mathcal{C}_{\mu\nu} := \mathcal{E}_{\mu\nu} - 8\pi G_{\mathrm{N}} T_{\mu\nu} = \frac{16\pi G_{\mathrm{N}}}{\sqrt{-g}}\frac{\delta\Gamma_{\mathrm{IR}}}{\delta g^{\mu\nu}}. \tag{22}$$

For an Einstein–Hilbert term, a cosmological term, representative local higher-curvature terms, and massless-loop corrections,

$$\mathcal{C}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} + H^{\mathrm{loc}}_{\mu\nu} + H^{\mathrm{nl}}_{\mu\nu} - 8\pi G_{\mathrm{N}} T_{\mu\nu}. \tag{23}$$

The tensor H^{nl} includes the variation of nonlocal form factors, not merely a running local coefficient. Boundary terms required for the chosen variational problem are included in Γ_{IR}.

For a Wilsonian action at finite derivative order, write

$$\mathcal{C}^{\mathrm{RG}}_{\mu\nu} := \mathcal{C}^{\mathrm{loc}}_{\mu\nu}. \tag{24}$$

This is the defect used by the local RG theorem below. The full one-particle-irreducible defect is

$$\mathcal{C}_{\mu\nu} = \mathcal{C}^{\mathrm{loc}}_{\mu\nu} + \mathcal{C}^{\mathrm{nl}}_{\mu\nu}. \tag{25}$$

A local beta functional cannot generically encode the second term, whose value at one point depends on fields elsewhere. A full nonlocal extension is stated separately after the main theorem. The covector $\mathbf{C}^{\mathrm{RG}}_g$ is obtained from equation (26) by replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{RG}}$.

Define the defect covector on a metric variation $h^{\mu\nu}$ by

$$\mathbf{C}_g(h) := \frac{1}{16\pi G_{\mathrm{N}}} \int_M \mathrm{d}^d x \sqrt{-g}\, \mathcal{C}_{\mu\nu} h^{\mu\nu}. \tag{26}$$

On matter shell and with the specified boundary conditions,

$$\delta_g \Gamma_{\mathrm{IR}}[h] = \mathbf{C}_g(h). \tag{27}$$

This covector, rather than a bare component expression, is the object that transforms naturally under field redefinitions.

Assumption 4.1 (Ward compatibility)**.** The full effective action and measure are invariant under diffeomorphisms, all retained matter and additional gravitational field equations hold, background sources contribute no Ward force, and there is no uncancelled diffeomorphism anomaly or boundary flux. Consequently,

$$\nabla^\mu \mathcal{C}_{\mu\nu} = 0. \tag{28}$$

Lemma 4.2 (Descent to the gauge quotient)**.** *Under theorem 4.1, the covector equation (26) annihilates compactly supported diffeomorphism directions and therefore descends to the regular quotient $\mathcal{Q} = \mathcal{F}/\mathrm{Diff}(M)$.*

Proof. For $h^{\mu\nu} = \mathcal{L}_\xi g^{\mu\nu} = -2\nabla^{(\mu}\xi^{\nu)}$, integration by parts gives

$$\mathbf{C}_g(\mathcal{L}_\xi g^{-1}) = \frac{1}{8\pi G_{\mathrm{N}}} \int_M \mathrm{d}^d x \sqrt{-g}\, \xi^\nu \nabla^\mu \mathcal{C}_{\mu\nu}, \tag{29}$$

up to the excluded boundary flux. The integral vanishes by equation (28). □

Finite local counterterms can move terms between the gravitational and matter pieces. If

$$\Gamma_{\mathrm{g}} \mapsto \Gamma_{\mathrm{g}} + I_{\mathrm{ct}}, \qquad \Gamma_{\mathrm{m}} \mapsto \Gamma_{\mathrm{m}} - I_{\mathrm{ct}}, \tag{30}$$

then the separate $\mathcal{E}_{\mu\nu}$ and $T_{\mu\nu}$ change but the full defect does not. A beta-functional relation must therefore use the complete renormalized defect, not an arbitrary gravity–matter split.

5 Field redefinitions and redundant operators

Let Ψ^A be coordinates on the full field and coupling space. A μ-independent invertible local change of coordinates is

$$\Psi'^A = F^A(\Psi), \qquad J^A{}_B = \frac{\delta F^A}{\delta \Psi^B}. \tag{31}$$

The beta functional is a vector,

$$\beta'^A = J^A{}_B \beta^B, \tag{32}$$

while the variational derivative is a covector,

$$\frac{\delta\Gamma}{\delta\Psi'^A} = (J^{-1})^B{}_A \frac{\delta\Gamma}{\delta\Psi^B}. \tag{33}$$

Thus an equation relating beta and gradient requires a bundle map from cotangent to tangent space.

If the redefinition depends explicitly on scale,

$$\Psi'^A = F^A_t(\Psi), \tag{34}$$

then

$$\beta'^A = J^A{}_B\beta^B + \partial_t F^A_t\big|_\Psi . \tag{35}$$

The second term can make a component beta nonzero at a point where the original components vanished. A connection is precisely what separates this chosen motion of coordinates from horizontal physical running.

Definition 5.1 (Redundant direction)**.** A local tangent vector $R^A_\alpha\varepsilon^\alpha$ is redundant if its infinitesimal action is induced by an invertible local change of fields, possibly together with a gauge transformation, and leaves the physical observables invariant after all induced coupling changes are included.

At the action level, an infinitesimal local redefinition $\Psi^A \mapsto \Psi^A + \varepsilon F^A$ changes the action by

$$\delta_F S = \int \mathrm{d}^d x\, F^A(x)\frac{\delta S}{\delta\Psi^A(x)} \tag{36}$$

up to a boundary term and the quantum Jacobian. Operators proportional to lower-order equations of motion are therefore redundant in the usual effective-theory sense. The equivalence theorem identifies the on-shell content preserved by suitable local field redefinitions [6].

For a metric effective theory, consider

$$g_{\mu\nu} \mapsto g_{\mu\nu} + a\ell_*^2 R_{\mu\nu} + b\ell_*^2 R g_{\mu\nu}. \tag{37}$$

Substituting equation (37) into the Einstein–Hilbert action moves coefficients among $R_{\mu\nu}R^{\mu\nu}$, R^2, and matter-curvature operators at order ℓ_*^2. Therefore the beta function of an individual curvature-squared coefficient is not invariant. The physical quotient flow and on-shell amplitudes are the invariant content.

Proposition 5.2 (Field-coordinate covariance)**.** *Suppose a beta-gradient relation has the coordinate form*

$$\beta^A = -\mathcal{M}^{AB}\partial_B\mathcal{A}. \tag{38}$$

Under an invertible scale-independent field redefinition, it retains this form provided

$$\mathcal{M}'^{AB} = J^A{}_C J^B{}_D \mathcal{M}^{CD}, \qquad \partial'_A\mathcal{A} = (J^{-1})^B{}_A\partial_B\mathcal{A}. \tag{39}$$

Proof. Use equation (32) and substitute equation (38):

$$\begin{aligned}\beta'^A &= -J^A{}_C\mathcal{M}^{CD}\partial_D\mathcal{A}\\ &= -J^A{}_C\mathcal{M}^{CD}J^B{}_D\partial'_B\mathcal{A} = -\mathcal{M}'^{AB}\partial'_B\mathcal{A}.\end{aligned} \tag{40}$$

The functional statement follows with integral kernels in place of matrices. □

The proposition fails in this simple form for explicitly scale-dependent coordinates because of equation (35). One must transform the comparison connection as well. This is why a metric beta functional cannot be defined by differentiating components without specifying the horizontal structure.

6 Scheme dependence and physical fixed points

A renormalization scheme selects coordinates on theory space, subtraction conditions, and finite local counterterms. Consider an analytic finite reparametrization of dimensionless couplings,

$$c'^I = f^I(c), \qquad J^I{}_J = \frac{\partial f^I}{\partial c^J}. \tag{41}$$

For a scale-independent regular transformation,

$$\beta'^I(c') = J^I{}_J(c)\beta^J(c). \tag{42}$$

Individual coefficients in a perturbative beta function can change, but a zero is carried to a zero if J is nonsingular.

Proposition 6.1 (Fixed-point covariance)**.** *Let c_* be a fixed point and let equation (41) be invertible in a neighborhood of c_*. Then $c'_* = f(c_*)$ is a fixed point. Moreover, the linear stability matrices at the two points are similar:*

$$B'^I{}_J(c'_*) = J^I{}_K(c_*)B^K{}_L(c_*)(J^{-1})^L{}_J(c'_*), \tag{43}$$

where $B^I{}_J = \partial_J\beta^I$.

Proof. Equation equation (42) gives $\beta'(c'_*) = J(c_*)\beta(c_*) = 0$. Differentiate that equation with respect to c'. A term containing ∂J is proportional to $\beta(c_*)$ and vanishes. The remaining term is equation (43). □

The eigenvalues of B are therefore invariant under regular scale-independent reparametrizations. Their interpretation as critical exponents also requires removal of redundant directions. Singular changes of variables, truncations, and explicitly scale-dependent transformations can change the apparent fixed locus and are not covered by theorem 6.1.

For local functionals, a scheme change can include

$$\Gamma \mapsto \Gamma + I_{\text{finite}}[g, \Phi; c]. \tag{44}$$

This changes local anomaly coefficients and beta representatives. It can also shift the displayed defect if the counterterm changes the total action rather than merely transferring a term between sectors. Once physical renormalization conditions are fixed, the beta-gradient relation must use the defect of that same renormalized functional. Combining a beta from one scheme with an Euler derivative from another has no invariant meaning.

Definition 6.2 (Physical fixed point)**.** A reconstruction is at a physical fixed point if its horizontal beta vector is redundant:

$$[\widehat{\beta}] = 0 \quad \text{in} \quad T\mathcal{F}/\mathcal{V}. \tag{45}$$

For the metric component this allows $\widehat{\beta}_g = \mathcal{L}_\xi g^{-1}$ and other admitted redundant field directions, not only the component equation $\widehat{\beta}_g = 0$.

The definition follows the general RG lesson that redundant operators generate changes of description rather than new physics [5]. It is particularly important in the sigma model, where target-space diffeomorphisms change the metric beta representative.

7 Local RG on reconstruction space

The standard local RG equation equation (16) concerns a QFT with local sources. To speak of a local RG for reconstructed geometry, one must supply an analogous generator on the reconstruction field bundle. We now state the minimum structure used later.

Let $\sigma(x)$ be a smooth local scale parameter. On a regular patch of reconstruction space, define

$$\begin{aligned}\Delta^{\text{rec}}_\sigma = \int_M \mathrm{d}^d x\sqrt{-g}\bigg[&\sigma\widehat{\beta}^{\mu\nu}_g \frac{\delta}{\delta g^{\mu\nu}} + \sigma\widehat{\beta}^A_\Phi \frac{\delta}{\delta\Phi^A} + \sigma\widehat{\beta}^I \frac{\delta}{\delta c^I} \\ &+ (\nabla_\mu\sigma)\mathcal{J}^\mu + (\nabla_\mu\nabla_\nu\sigma)\mathcal{K}^{\mu\nu} + \cdots\bigg].\end{aligned} \tag{46}$$

Derivative-of-σ terms are allowed because local counterterms and operator mixing generate them. The generator is not determined by the global flow $\mathcal{D}_t$ alone.

Assumption 7.1 (Local reconstruction RG)**.** There is a local covariant generator equation (46) satisfying:

(i) its constant-σ restriction agrees with the comparison connection defining the global reconstruction flow;

(ii) its commutator closes modulo diffeomorphisms, redundant field transformations, and a stated anomaly:

$$[\Delta^{\text{rec}}_{\sigma_1}, \Delta^{\text{rec}}_{\sigma_2}] \equiv 0 \mod \mathcal{V}; \tag{47}$$

(iii) thresholds and changes of field content are excluded from the open scale interval or handled by explicit matching conditions;

(iv) the beta functionals are local through the derivative order retained in the infrared expansion.

This is a strong hypothesis. A family of smoothed metrics can be differentiable in one global scale without admitting arbitrary local scale transformations. Even when theorem 7.1 holds, the Wess–Zumino-type condition equation (47) constrains beta data but does not automatically produce a variational potential.

Definition 7.2 (Variational beta structure)**.** Let $\mathbf{C}_g^{\mathrm{RG}}$ be the covector of the local Wilsonian defect equation (24) on the physical metric quotient. A variational beta structure on an open set $\mathcal{U} \subset \mathcal{Q}$ is a local bundle map

$$\mathcal{M}_g : T_g^*\mathcal{Q} \longrightarrow T_g\mathcal{Q} \tag{48}$$

such that

$$[\widehat{\beta}_g] = -\mathcal{M}_g[\mathbf{C}_g^{\mathrm{RG}}] \tag{49}$$

after the retained nonmetric beta directions are either set to their own fixed conditions or included in a specified block extension of equation (49).

In a local coordinate representation,

$$\begin{aligned}\widehat{\beta}_g^{\mu\nu}(x) = -\int_M \mathrm{d}^d y\sqrt{-g(y)}\,\mathcal{M}^{\mu\nu|\rho\sigma}(x,y)\frac{\mathcal{C}^{\mathrm{RG}}_{\rho\sigma}(y)}{16\pi G_{\mathrm{N}}}\\ + \mathcal{L}_\xi g^{\mu\nu}(x) + R^{\mu\nu}_{\mathrm{red}}(x).\end{aligned} \tag{50}$$

Locality means that the kernel is supported on the diagonal and is a finite differential operator to the stated order. Dimensional factors needed to turn curvature into a dimensionless t derivative are contained in $\mathcal{M}$.

Assumption 7.3 (Nondegeneracy)**.** The induced map $\mathcal{M}_g$ is injective on the admissible physical defect covectors. Equivalently,

$$\mathcal{M}_g\mathbf{C}_g^{\mathrm{RG}} = 0 \quad \Longrightarrow \quad \mathbf{C}_g^{\mathrm{RG}} = 0 \tag{51}$$

on the regular quotient with the stated boundary conditions.

If $\mathcal{M}_g$ is a differential operator, injectivity includes the absence of zero modes in the admissible domain. Pointwise nonzero coefficients are not enough. Boundary conditions and gauge projection can determine whether a kernel exists.

Remark 7.4 (Mixing with other beta functions). If metric, matter, and coupling directions mix, the correct relation may be

$$\begin{pmatrix}[\widehat{\beta}_g]\\ [\widehat{\beta}_\Phi]\\ \widehat{\beta}^I\end{pmatrix} = -\mathbb{M}\begin{pmatrix}[\mathbf{C}_g^{\mathrm{RG}}]\\ [\mathbf{E}_\Phi]\\ \partial_I\mathcal{A}\end{pmatrix}. \tag{52}$$

Then simultaneous fixed-point equivalence requires nondegeneracy of the full block map $\mathbb{M}$. A metric row alone need not determine the metric defect when off-diagonal mixing is present.

8 Conditional metric beta-functional theorem

We can now state the precise result. Its hypotheses separate the existence of a flow from its relation to the metric equation.

Theorem 8.1 (Conditional metric beta-functional equivalence)**.** *Let $\mathfrak{R}_\mu$ be a scale-indexed reconstruction on a regular open set and assume:*

(i) the reconstruction comparison structure of theorem 2.1 defines a horizontal metric beta functional modulo redundant directions;

(ii) the local reconstruction RG hypothesis theorem 7.1 holds;

(iii) the local Wilsonian infrared functional has a well-posed variational principle, the retained field equations hold, and the anomaly, source, and boundary hypotheses of theorem 4.1 hold, so $\mathbf{C}_g^{\mathrm{RG}}$ descends to the physical quotient;

(iv) all nonmetric fixed conditions required to remove mixing have been imposed, or the full block relation equation (52) is used;

(v) the variational relation equation (49) holds with the same renormalized functional and scheme that define $\mathbf{C}_g^{\mathrm{RG}}$;

(vi) the mobility map is injective on the admissible physical defect space.

Then

$$[\widehat{\beta}_g] = 0 \quad \Longleftrightarrow \quad [\mathbf{C}_g^{\mathrm{RG}}] = 0. \tag{53}$$

If reconstruction-tangent completeness also holds, the second condition is equivalent to the full local tensor equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + H_{\mu\nu}^{\mathrm{loc}} = 8\pi G_{\mathrm{N}} T_{\mu\nu}. \tag{54}$$

Without tangent completeness, both statements are restricted to the projected defect tested by admissible reconstruction variations.

Proof. Hypotheses (i)–(iii) place both sides of equation (49) on the same physical tangent-cotangent pair over the reconstruction quotient. Hypothesis (iv) removes untracked off-diagonal source terms. If $[\mathbf{C}_g^{\mathrm{RG}}] = 0$, the variational relation immediately gives $[\widehat{\beta}_g] = 0$. Conversely, if $[\widehat{\beta}_g] = 0$, then

$$\mathcal{M}_g[\mathbf{C}_g^{\mathrm{RG}}] = 0. \tag{55}$$

Injectivity from hypothesis (vi) gives $[\mathbf{C}_g^{\mathrm{RG}}] = 0$. Ward compatibility ensures that this statement is independent of the diffeomorphism representative. When admissible reconstruction variations span all local metric variations modulo gauge, the fundamental lemma of the calculus of variations converts the vanishing defect covector into the pointwise tensor equation equation (54). Without that spanning condition, only the pullback covector vanishes. □

Corollary 8.2 (Leading infrared equation)**.** *Add the infrared hypotheses of a local diffeomorphism-invariant derivative expansion, one interacting massless metric spin-two mode, no other unsuppressed long-range geometric fields, controlled Wilson coefficients, and the appropriate dimension. Then a physical metric fixed point satisfying the hypotheses of theorem 8.1 obeys at leading local two-derivative order*

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}, \tag{56}$$

with controlled local corrections retained as in equation (54). Massless nonlocal terms belong to the full one-particle-irreducible equation and are not fixed by this local RG corollary.

Proof. The theorem gives the local Wilsonian defect equation. The stated effective-field theory assumptions identify Einstein–Hilbert plus the cosmological operator as the leading local metric terms. They do not remove nonlocal form factors from retained massless fields. This is the derivative-order argument, not a new consequence of the beta functional. □

The order of logic matters. The beta-functional theorem does not derive the Einstein–Hilbert action. It first requires an effective action and its defect. The separate infrared classification then identifies the leading local terms.

Corollary 8.3 (Nonlocal extension)**.** *Drop the locality requirement on the reconstruction generator and mobility. If a specified, injective, possibly nonlocal bundle map instead obeys*

$$[\widehat{\beta}_g^{\mathrm{full}}] = -\mathcal{M}_g^{\mathrm{full}}[\mathbf{C}_g], \tag{57}$$

then its fixed points are equivalent to the full equation containing $H_{\mu\nu}^{\mathrm{nl}}$.

Proof. The two implications use the relation and injectivity exactly as in theorem 8.1. The result is not a local RG theorem because its generator or mobility can contain long-range kernels. □

8.1 When a monotonicity statement follows

Suppose there is a functional $\mathcal{A}[g]$ with

$$\delta\mathcal{A}[h] = \mathbf{C}_g^{\mathrm{RG}}(h) \tag{58}$$

after explicit scale dependence, other source variations, and boundary fluxes have been removed. If $\mathcal{M}_g$ is symmetric and positive on a gauge-fixed physical Euclidean field space, then along $\widehat{\beta}_g = -\mathcal{M}_g \mathbf{C}_g^{\mathrm{RG}}$,

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} = -\langle \mathbf{C}_g^{\mathrm{RG}}, \mathcal{M}_g \mathbf{C}_g^{\mathrm{RG}} \rangle \le 0. \tag{59}$$

Proposition 8.4 (Strict gradient monotonicity)**.** *Under the conditions above, if $\mathcal{M}_g$ is strictly positive on nonzero physical covectors, equality in equation (59) holds exactly at a defect-free fixed point.*

Proof. The chain rule and equation (58) give $\mathrm{d}\mathcal{A}/\mathrm{d}t = \mathbf{C}_g^{\mathrm{RG}}(\widehat{\beta}_g)$. Substitute the gradient relation. Positivity gives the inequality, and strict positivity gives the equality condition. ☐

This proposition is not automatic in Lorentzian gravity. The conformal mode and gauge directions obstruct a naive positive metric on the full space of Lorentzian metrics. Analytic continuation, gauge fixing, boundary conditions, and projection to physical modes must be justified before calling $\mathcal{A}$ monotone. Fixed-point equivalence in theorem 8.1 needs injectivity but not positivity.

8.2 Necessity of each structural hypothesis

If no comparison connection is supplied, $\widehat{\beta}_g$ is coordinate dependent. If no local RG structure exists, a global smoothing derivative cannot be promoted to a local beta functional. If the Ward identity fails, the defect does not descend to the diffeomorphism quotient. If the variational relation is absent, beta zeros and defect zeros are unrelated subsets. If $\mathcal{M}_g$ has a kernel, a nonzero defect can map to zero. If tangent completeness fails, a vanishing pulled-back covector tests only a projection. The examples below realize these failures explicitly.

9 Counterexamples without the required structure

9.1 Running couplings on an off-shell metric

Let

$$\Gamma_k[g] = \frac{1}{16\pi G(k)} \int \sqrt{-g}(R - 2\Lambda(k)) + c_1(k) \int \sqrt{-g} R^2. \tag{60}$$

Choose an arbitrary metric g_0 that does not solve its Euler equation, and define the Wilsonian trajectory by running only G, Λ, and c_1 while holding the metric argument fixed. Then

$$\widehat{\beta}_g[g_0] = 0, \qquad \mathcal{C}_{\mu\nu}[g_0] \neq 0. \tag{61}$$

This is not inconsistent. Wilsonian coupling flow and field stationarity are different operations. It is a direct counterexample to the universal identity equation (1).

9.2 A singular mobility

Consider a two-coordinate truncation with defect covector

$$\mathbf{C} = (C_1, C_2) \tag{62}$$

and mobility

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{63}$$

The relation $\beta = -\mathcal{M}\mathbf{C}$ gives $\beta = 0$ for $\mathbf{C} = (0, 1)$. Thus a displayed gradient relation is not enough; nondegeneracy on the admissible physical subspace is essential.

9.3 Closed flow without a gradient potential

On $\mathbb{R}^2 \setminus \{0\}$ let

$$\beta(x, y) = (-y, x). \tag{64}$$

Its trajectories are closed circles. Suppose it were a negative gradient flow of a single-valued potential with a positive mobility. The potential would strictly decrease along every nonstationary orbit by equation (59), but it must return to the same value after one period. This contradiction shows that no such positive gradient structure exists.

Now choose

$$\mathcal{A}(x, y) = \frac{1}{4}(x^2 + y^2 - 1)^2. \tag{65}$$

Its defect $\mathrm{d}\mathcal{A}$ vanishes on the unit circle, while equation (64) is nonzero there. The defect-free set and the beta fixed set disagree because no beta-gradient relation was imposed.

9.4 Scale-dependent field coordinates

Let $x' = x + t$ in a one-dimensional field space. A stationary trajectory $\beta_x = 0$ becomes

$$\beta_{x'} = 1 \tag{66}$$

by equation (35). Conversely, a nonzero beta can be cancelled by a moving coordinate. Raw beta components therefore do not define a scheme-independent fixed point under scale-dependent reparametrizations. The comparison connection must transform with the coordinates.

9.5 Pure diffeomorphism flow

Equation equation (8) gives a nonzero tensor beta representative with zero physical flow. Equating its components to the defect would demand that a gauge artifact equal a horizontal variational tensor. The correct statement is made only after quotienting both sides.

9.6 Incomplete reconstruction tangent

Suppose reconstruction variations are purely conformal,

$$\delta g^{\mu\nu} = -2\delta\omega\, g^{\mu\nu}. \tag{67}$$

Stationarity tests only $g^{\mu\nu}\mathcal{C}_{\mu\nu} = 0$. A traceless nonzero defect is invisible. A metric beta defined only along this conformal family can at most reproduce the trace equation, even if its one-dimensional mobility is nondegenerate. Tangent completeness is logically separate from gradient nondegeneracy.

10 Fixed-point examples

10.1 A finite positive gradient system

Let

$$\mathcal{A}(x, y) = \frac{1}{2}x^2 + 2y^2, \qquad \mathbf{C} = (x, 4y), \qquad \mathcal{M} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{68}$$

Then

$$\beta = -\mathcal{M}\mathbf{C} = (-2x, -2y), \tag{69}$$

so the unique fixed point is the unique defect-free point $(0, 0)$. Moreover,

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} = -2x^2 - 8y^2 < 0 \tag{70}$$

away from the origin. This example contains exactly the algebra of theorems 8.1 and 8.4 and none of the continuum physics.

10.2 A redundant fixed trajectory

Let (x, y) have a redundancy $y \sim y + s$. The beta

$$\beta = (0, 1) \tag{71}$$

is nonzero in components but zero in the quotient, whose physical coordinate is x. If the defect covector is $(x, 0)$, the quotient relation with mobility 1 gives $[\beta_x] = -[x]$. The physical fixed point is $x = 0$ regardless of the arbitrary motion in y.

10.3 Ricci-flat sigma-model backgrounds

For a metric-only two-dimensional sigma model with no B field and constant dilaton, the leading target metric beta function has the form

$$\beta^g_{ij} = \alpha' R_{ij} + \mathcal{O}(\alpha'^2) \tag{72}$$

up to the sign convention for RG time. A flat torus satisfies $R_{ij} = 0$ and is a one-loop fixed background. At this order it also solves the vacuum target-space Einstein equation. This agreement is real but domain-specific: $g_{ij}(X)$ is a world-sheet coupling and the target-space action relation is supplied by sigma-model Weyl consistency.

10.4 Einstein space with cosmological term

Let a target metric satisfy

$$R_{ij} = \lambda g_{ij}, \qquad \lambda \neq 0. \tag{73}$$

It can solve a spacetime Einstein equation with an appropriate cosmological constant. But the metric-only one-loop sigma beta equation (72) is nonzero. Additional backgrounds, a dilaton, central-charge balance, or a modified fixed-point condition are required. This example shows that even the canonical sigma-model analogy is not the raw identity beta equals the general Einstein defect.

10.5 A fixed theory with an off-shell family

At a coupling fixed point $\beta^I(c_*) = 0$, the generating functional remains a functional of arbitrary background sources. Only its stationary points solve field equations. Thus an RG fixed theory contains many off-shell metric arguments. “The theory is at a fixed point” and “this metric is on shell” are statements about different spaces.

11 Nonlinear sigma models and string backgrounds

The sigma model is the established case in which a metric is literally a running coupling. It is therefore the right comparison, provided its domain is kept explicit.

11.1 The target metric as a world-sheet coupling

For a Euclidean world sheet (Σ, h) and maps $X : \Sigma \to \mathcal{T}$ into a target manifold, write schematically

$$\begin{aligned} S_\sigma[X; g, B, \Phi] = \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2 z \sqrt{h} \Big[& h^{ab} g_{ij}(X) \partial_a X^i \partial_b X^j \\ & + i\epsilon^{ab} B_{ij}(X) \partial_a X^i \partial_b X^j + \alpha' \Phi(X) R^{(2)}(h) \Big]. \end{aligned} \tag{74}$$

The functions $g_{ij}(X)$, $B_{ij}(X)$, and $\Phi(X)$ are infinitely many couplings of a two-dimensional QFT. Renormalizing that QFT produces beta functionals on the space of target backgrounds. Friedan’s analysis makes the RG action on the space of Riemannian target metrics explicit [10, 11].

Let

$$H_{ijk} = 3\partial_{[i}B_{jk]}. \tag{75}$$

In a standard leading-order scheme, the metric and antisymmetric-tensor beta representatives have the schematic form

$$\beta^g_{ij} = \alpha' \left(R_{ij} - \frac{1}{4} H_{ikl} H_j{}^{kl} \right) + \mathcal{O}(\alpha'^2), \tag{76}$$

$$\beta^B_{ij} = -\frac{\alpha'}{2} \nabla^k H_{kij} + \mathcal{O}(\alpha'^2). \tag{77}$$

The sign changes if the RG time is oriented from ultraviolet to infrared rather than with increasing subtraction scale. Fixed points are unaffected by that overall convention.

The coefficients controlling the world-sheet trace anomaly are not always the raw beta representatives. Up to conventions and redundant target diffeomorphisms, the leading Weyl coefficient for the metric is

$$\bar{\beta}^g_{ij} = \alpha' \left(R_{ij} - \frac{1}{4} H_{ikl} H_j{}^{kl} + 2\nabla_i \nabla_j \Phi \right) + \mathcal{O}(\alpha'^2). \tag{78}$$

There are companion equations for B and Φ. The dilaton term in equation (78) can be viewed partly as a diffeomorphism improvement of the raw metric beta, but the coupled Weyl-invariance conditions also contain physical dilaton information. One must not keep equation (78) while discarding the dilaton equation.

11.2 Background equations and a target-space action

World-sheet Weyl invariance requires the appropriate anomaly coefficients to vanish. At leading order, these equations arise from a target-space action of the form

$$S_{\text{target}} = \frac{1}{2\kappa^2} \int \mathrm{d}^D X \sqrt{|g|}\, e^{-2\Phi} \left[R + 4(\nabla\Phi)^2 - \frac{1}{12} H_{ijk} H^{ijk} - 2\Lambda_{\text{c}} + \mathcal{O}(\alpha') \right], \tag{79}$$

where Λ_{c} represents the central-charge deficit in the noncritical theory. Callan, Friedan, Martinec, and Perry established the background-field relation between conformal invariance and spacetime equations [12]. Subsequent analyses clarified the relation among Weyl coefficients, target-space actions, and consistency conditions [13, 14, 15, 16, 17].

The relation is not a literal equality of the form $\bar{\beta}^g_{ij} = \delta S_{\text{target}}/\delta g^{ij}$. The dilaton measure, normalization, mixing among background equations, field redefinitions, and a field-space metric intervene. In the notation of theorem 7.2, sigma-model calculations supply a concrete version of the bundle map $\mathcal{M}$ and the coupled block structure.

Proposition 11.1 (Sigma-model fixed-point statement)**.** *Within a sigma-model scheme and derivative expansion for which the coupled Weyl coefficients are related by a nondegenerate field-space map to the variations of equation (79), simultaneous vanishing of the physical Weyl coefficients is equivalent, modulo target-space field redefinitions, to the coupled target-background Euler equations to that order.*

Proof. The background couplings form the coordinates $\Psi^A = (g, B, \Phi, \ldots)$. By hypothesis their physical Weyl coefficients and the action variations obey the nondegenerate block relation equation (52). The proof is then the block version of theorem 8.1: one implication follows directly from the relation and the converse follows from injectivity. Target-space diffeomorphisms and gauge transformations are removed as redundant directions. ☐

The proposition intentionally includes its scheme and derivative-order qualifications. Higher-loop coefficients can be moved by local redefinitions, while the physical conformal backgrounds are invariant under regular changes of description.

11.3 The Friedan fixed-point equation

In $2+\epsilon$ dimensions, canonical scaling contributes a metric term to the fixed-point equation. Friedan's formulation includes equations of the form

$$R_{ij} - a g_{ij} = \nabla_i v_j + \nabla_j v_i \tag{80}$$

at leading order and in an appropriate normalization. The vector v_i represents a target diffeomorphism direction, and a depends on dimension and normalization. Equation equation (80) illustrates three lessons: fixed points are considered modulo redundancy, canonical scaling matters, and the beta equation need not be Ricci flatness outside the strictly two-dimensional metric-only case.

12 Precise analogy with reconstructed geometry

The analogy can now be stated without collapsing distinct theories. In the sigma model, the target metric is a world-sheet coupling. In a reconstruction, the metric is the output of $\mathfrak{R}_\mu$ and may also be a variational field of a spacetime effective action. These roles coincide only if a microscopic construction supplies the local RG and gradient data.

Structure	Sigma model or string background	Reconstructed geometry
Running object	$g_{ij}(X)$ is an infinite set of two-dimensional couplings	$g_{\mu\nu} = \pi_g \mathfrak{R}_\mu(q)$ is a scale-indexed reconstruction output
Scale generator	World-sheet QFT renormalization and Weyl transformation	Must be supplied by a comparison connection and local reconstruction RG
Redundancy	Target diffeomorphisms and background gauge transformations	Spacetime diffeomorphisms and admitted local field redefinitions
Variational functional	String target-space action or central-charge action in a specified scheme	Infrared spacetime effective action defining $\mathcal{C}_{\mu\nu}$
Fixed-point equation	Simultaneous vanishing of physical Weyl coefficients	Vanishing of $[\widehat{\beta}_g]$ plus other retained physical betas
Gradient relation	Derived perturbatively with field-space mixing and scheme qualifications	An additional hypothesis unless obtained from a microscopic reconstruction
Leading metric tensor	Ricci tensor plus H, dilaton, and higher-α' corrections	Einstein defect plus local EFT and massless nonlocal corrections

Table 1: The sigma-model analogy and the additional data required for a reconstructed metric.

The following statements are therefore justified:

(1) A metric can be an infinite-dimensional coupling with a tensor beta function. Sigma models prove this in their domain.

(2) Vanishing beta or Weyl coefficients can encode spacetime equations when a nondegenerate variational relation is established.

(3) Field redefinitions and gauge directions require quotient fixed points.

(4) The leading tensor depends on the action, field content, dimension, and expansion parameter of the theory being renormalized.

The following statement is not justified:

$$\text{every coarse-graining of quantum data} \quad \Longrightarrow \quad \beta_g \propto G - 8\pi G T. \tag{81}$$

In particular, the sigma-model scale is the world-sheet renormalization scale. The reconstruction scale may be a resolution in a microscopic algebra, a tensor network depth, an energy cutoff, or another parameter. An identification between the two scales would itself be a model-dependent dictionary.

13 Nonlocal terms and cancellation of scale dependence

Massless fields generate nonlocal terms in an infrared effective action. A representative four-dimensional expression is

$$\Gamma_\mu[g] \supset c_R(\mu) \int \mathrm{d}^4x \sqrt{-g}\, R^2 \\ + b_R \int \mathrm{d}^4x \sqrt{-g}\, R \log\left(\frac{-\Box - i0}{\mu^2}\right) R. \tag{82}$$

The explicit μ derivative of the logarithm can cancel the running of $c_R(\mu)$ in the complete functional. In the normalization of equation (82), scale independence of this pair gives schematically

$$\mu \frac{\mathrm{d}c_R}{\mathrm{d}\mu} = 2b_R, \tag{83}$$

with operator mixing added in a complete basis. This cancellation is a Callan–Symanzik statement. It neither sets the variation of equation (82) to zero nor makes g flow.

The nonlocal defect contains variations of the curvatures, the measure, and the operator $\log(-\Box/\mu^2)$. Its causal interpretation also depends on whether one uses an in-out or in-in effective action. Gravitational EFT separates analytic local coefficients from nonanalytic low-energy information carried by massless propagation [18].

Proposition 13.1 (Scale cancellation is not stationarity)**.** *There exist functionals $\Gamma_\mu[g]$ satisfying a complete Callan–Symanzik equation for every g while $\delta\Gamma_\mu/\delta g^{\mu\nu}$ is nonzero on an open set of metrics.*

Proof. Choose the coefficients in equation (82) to satisfy equation (83). The total scale derivative cancels at the stated order for all metric arguments. The metric Euler derivative of R^2 and of the nonlocal form factor is generically nonzero away from their stationary backgrounds. Thus scale cancellation and stationarity are independent functional equations. $\Box$

The locality assumption in theorem 7.1 can fail when the proposed beta functional itself contains long-range kernels. A nonlocal but injective relation may still give a formal zero-locus equivalence, but it is not the local RG theorem proved here. Conversely, truncating a nonlocal defect to finitely many local operators can create spurious fixed points.

14 Boundary conditions, signature, and zero modes

The defect covector and mobility map depend on their domains. For the Einstein–Hilbert action on a non-null boundary with Dirichlet metric data, the Gibbons–Hawking–York term removes normal derivatives of the metric variation. Null boundaries, corners, mixed data, and higher-curvature actions require their own completions. A reconstruction flow that changes boundary data is not tangent to the same variational problem.

Suppose $\mathcal{M}_g$ is a differential operator. Its injectivity can fail because of Killing vectors, moduli, boundary zero modes, or gauge directions. These modes must be quotiented or fixed before applying theorem 7.3. An inverse Green function, if used, depends on the same boundary prescription.

Lorentzian signature adds a separate issue. A hyperbolic differential operator does not define a positive mobility merely because it is invertible with chosen initial data. Retarded and advanced inverses encode different physical questions. Therefore theorem 8.1 uses injectivity for fixed-point equivalence and reserves positivity for the stronger monotonicity statement.

Example 14.1 (Modulus zero mode)**.** Let $g(a)$ be a family of flat torus metrics parametrized by a modulus a. The local Ricci defect vanishes along the family, so the action Hessian has a zero mode tangent to a before additional potentials or identifications are included. A proposed mobility derived from that Hessian is not strictly positive or invertible on the full moduli tangent. Fixed points can form a manifold rather than an isolated point.

15 Relation to the reconstruction consistency program

The present result begins after three logically prior steps. First, an infrared reconstruction must supply a smooth Lorentzian metric, locality, diffeomorphism redundancy, scale separation, a controlled derivative expansion, and appropriate spectrum and anomaly assumptions. Second, the reconstructed metric must be a variational field of the infrared action. Third, the defect equation (22) must be Ward compatible and the reconstruction tangent must be complete enough to test it.

Those steps imply a variational metric equation but no metric beta functional. The additional implications established here are

$$\begin{gathered}\text{scale-indexed reconstruction} + \text{comparison connection}\\ +\text{local reconstruction RG} + \text{nondegenerate variational beta relation}\\ \Longrightarrow \quad \big([\widehat{\beta}_g] = 0 \Longleftrightarrow [\mathbf{C}_g^{\mathrm{RG}}] = 0\big).\end{gathered} \tag{84}$$

None of the four added terms in the left side follows from the existence of an information metric alone.

If the effective action satisfies the one-massless-spin-two and controlled EFT hypotheses, the defect has Einstein form at leading local derivative order. The beta-functional route then reproduces that already classified leading equation. It does not independently establish the spectrum, compute Newton's constant, or construct the microscopic reconstruction.

The defect is invariant under transferring a finite local counterterm between the gravity and matter bookkeeping sectors. The beta representative can still change under a scheme transformation. A valid relation pairs both objects in one scheme and then uses theorem 5.2 to transport the entire structure to another regular field coordinate system.

16 Limitations and failure modes

The theorem has a deliberately narrow domain.

16.1 No universal reconstruction RG

A family of coarse-graining channels need not yield a differentiable flow of metrics. It may change topology, dimension, signature, or field content. Threshold matching can be piecewise smooth without defining one beta functional on a single bundle.

16.2 No automatic locality

Geometric smoothing is often nonlocal, and integrating out gapless modes produces nonlocal form factors. A globally defined scale derivative does not imply the local generator equation (46).

16.3 No automatic gradient relation

Wess–Zumino consistency constrains a local RG but does not in general identify its metric beta with the Euler derivative of a spacetime action. The rotational counterexample equation (64) shows that a smooth flow can lack any positive gradient potential.

16.4 No universal positive metric on metric space

The word "gradient" can hide an indefinite or degenerate field-space pairing. Gauge modes, the conformal mode, moduli, and Lorentzian signature must be handled explicitly. Without positivity there is no general monotone functional, even if fixed-point equivalence holds.

16.5 Additional light fields

Scalar-tensor, vector-tensor, torsional, and multiple-metric theories require a block beta system. Setting only the metric beta to zero can leave other equations unsatisfied. The string example itself demonstrates this through the B field and dilaton.

16.6 Anomalies

An uncancelled diffeomorphism anomaly prevents the defect from descending to the metric gauge quotient. A Weyl anomaly is different: it is the local term on the right side of a local RG equation and can carry physical central-charge data. The two anomalies must not be conflated.

16.7 Truncation artifacts

A finite operator truncation can introduce or remove apparent fixed points. Large Wilson coefficients can defeat derivative ordering, while nonlocal massless terms cannot be represented by a finite analytic basis. Scheme stability should be tested at the order claimed.

16.8 No microscopic relaxation law

Even if $\widehat{\beta}_g = -\mathcal{M}_g \mathbf{C}_g$, the relation defines a flow on reconstruction space. It does not prove that physical time evolution of the microscopic system relaxes according to that flow. RG time, reconstruction scale, and Lorentzian time are distinct parameters.

17 Conclusion

A metric beta functional can encode an infrared gravitational equation, but only after the relevant structures are supplied. The Wilsonian running of couplings, Callan–Symanzik scale independence, local Weyl identities, and an actual scale flow of reconstructed metrics answer different questions. The last requires a comparison connection across reconstruction scales and a quotient by diffeomorphisms and redundant field redefinitions.

The central theorem assumes a local reconstruction RG and an injective bundle map from the defect covector to the physical metric beta vector. Under that variational relation, beta fixed points and defect-free metrics coincide. Tangent completeness is still needed to turn a projected reconstruction condition into the full local tensor equation. A positive symmetric mobility adds monotonicity only after gauge, signature, explicit scale dependence, and other fields are controlled.

The nonlinear sigma model supplies a precise precedent. Its target metric is a world-sheet coupling, and the coupled Weyl coefficients can be related to target-space background equations. This precedent motivates the form of the conditional theorem and simultaneously displays its necessary qualifications: field redefinitions, dilaton and antisymmetric-tensor mixing, scheme dependence, and derivative corrections. Without an analogous reconstruction dictionary, there is no universal equation $\beta_g = G - 8\pi G T$.

A Finite-dimensional form of the theorem

Let V be a finite-dimensional vector space, $A : V \to \mathbb{R}$ a smooth function, and $C = \mathrm{d}A \in V^*$. Let $M : V^* \to V$ be linear and set

$$\beta = -MC. \tag{85}$$

The finite theorem is immediate:

$$M \text{ injective} \quad \Longrightarrow \quad \big(\beta = 0 \Longleftrightarrow C = 0\big). \tag{86}$$

If M is represented by a square matrix, injectivity is $\det M \neq 0$. If M is symmetric positive definite,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -C^{\mathsf{T}} M C < 0 \tag{87}$$

away from a critical point.

For a redundant subspace $R \subset V$, replace V by V/R. A convenient computational representation chooses a projection $P : V \to V_{\text{phys}}$. Then a physical fixed point satisfies

$$P\beta = 0. \tag{88}$$

The relation reproduces a defect equation only if the induced map from the physical defect covectors to V_{phys} is injective. A nonsingular matrix on the unreduced coordinates can still fail after a poorly chosen projection, and a singular unreduced matrix can become nondegenerate after pure gauge zero modes are removed.

Under $x' = F(x)$ with Jacobian J, the transformations are

$$\beta' = J\beta, \qquad C' = J^{-\mathsf{T}}C, \qquad M' = JMJ^{\mathsf{T}}. \tag{89}$$

These are exactly the finite counterparts of equations (32), (33) and (39).

B Functional covariance with integral kernels

Use condensed indices $A = (\mu\nu, x)$ and $B = (\rho\sigma, y)$. A local field redefinition has functional Jacobian

$$J^A{}_B = \frac{\delta g'^A}{\delta g^B}. \tag{90}$$

The beta and defect covector transform as

$$\beta'^A = \int \mathrm{d}B\, J^A{}_B \beta^B, \tag{91}$$

$$C'_A = \int \mathrm{d}B\, (J^{-1})^B{}_A C_B. \tag{92}$$

If

$$\beta^A = -\int \mathrm{d}B\, M^{AB} C_B, \tag{93}$$

then covariance requires

$$M'^{AB} = \int \mathrm{d}C\, \mathrm{d}D\, J^A{}_C M^{CD} J^B{}_D. \tag{94}$$

Substitution verifies $\beta' = -M'C'$.

For metric variables the functional Jacobian also acts on tensor indices and delta distributions. If the redefinition contains derivatives, integrations by parts and boundary conditions enter the adjoint. A scale-dependent redefinition adds the inhomogeneous term in equation (35); the connection must transform so that the horizontal beta obeys equation (91).

C Diagnostic Haskell model

The companion Haskell modules implement exact rational arithmetic for the finite-dimensional statements in section A. They contain matrix-vector multiplication, a determinant test for nondegeneracy, beta pushforward under a scheme Jacobian, an explicitly scale-dependent reparametrization term, and a projection that removes a redundant direction.

The executable checks include:

(i) a nonsingular diagonal mobility for which beta zeros and defect zeros coincide over a finite sample;

(ii) the singular counterexample equation (63);

(iii) preservation of a zero beta under invertible scale-independent pushforward;

(iv) failure of raw fixed-point covariance under an explicitly scale-dependent coordinate;

(v) a redundant translation removed by a physical projection;

(vi) the rotational flow equation (64), which is tangent to circles rather than descending a radial potential.

These checks are diagnostics of the algebra and counterexamples. They are not proofs of locality, Ward identities, continuum functional analysis, or the existence of a reconstruction RG for any microscopic quantum system.

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