Reconstruction Defects, Diffeomorphism Ward Identities, and Conservation

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Abstract

Let a coarse-grained quantum system reconstruct a Lorentzian metric and let a specified diffeomorphism-invariant infrared effective action govern the reconstructed fields. The local reconstruction defect is the normalized metric Euler residual $\mathcal{C}_{\mu\nu} = \mathcal{E}_{\mu\nu} - 8\pi G_{\mathrm{N}} T_{\mu\nu}$. This paper derives its Ward identities without using the contracted Bianchi identity as a substitute for the matter equations. For a pure metric gravitational action, diffeomorphism invariance gives $\nabla^\mu \mathcal{E}_{\mu\nu} = 0$ off shell. The matter action instead gives $\nabla^\mu T_{\mu\nu} = 0$ only after the dynamical matter equations hold and only in the absence of momentum exchange with spacetime-dependent background sources, boundary flux, or a diffeomorphism anomaly. Extra gravitational fields add their own Euler terms to the gravitational identity. Under the corresponding hypotheses, the defect is covariantly conserved even away from the metric equation $\mathcal{C}_{\mu\nu} = 0$.

We prove that the defect transforms naturally under pullback, provided the action, reconstruction dictionary, background sources, and boundary data are all transformed. We also distinguish the bulk Noether identity from covariant-phase-space current and charge statements, for which boundary completion is essential. Several counterexamples expose common misidentifications: a nonzero tensor proportional to the metric is divergence-free; an external scalar source produces a force-density term; an off-shell gravitational scalar obstructs the pure metric identity; and a nonlocal effective action need not have a local defect. Finally, the defect does not determine a microscopic relaxation law. Such a law additionally requires a time variable, a kinetic or mobility structure, causal data, and a choice of lift through the reconstruction map. The same Euler residual is compatible with inequivalent gradient, Hamiltonian, hyperbolic, nonlocal, and stochastic evolutions. Conservation is therefore a compatibility condition on the reconstructed field equation, not a unique dynamics for the microscopic system.

Contents

1 Introduction

A reconstructed metric can fail to satisfy its infrared field equation. Once an effective action has been specified, that failure is measured by an Euler residual. For the normalization used throughout this series, the residual is

$$\mathcal{C}_{\mu\nu} := \mathcal{E}_{\mu\nu} - 8\pi G_{\mathrm{N}} T_{\mu\nu}, \tag{1}$$

where $\mathcal{E}_{\mu\nu}$ is the normalized gravitational metric derivative and $T_{\mu\nu}$ is the matter stress tensor. The equation of motion is $\mathcal{C}_{\mu\nu} = 0$. The present question is different: what identity does $\mathcal{C}_{\mu\nu}$ obey because the action is diffeomorphism invariant?

The tempting answer is that the Bianchi identity implies conservation. That sentence merges three statements with different hypotheses. The contracted Bianchi identity,

$$\nabla^{\mu} G_{\mu\nu} = 0, \tag{2}$$

is a geometric identity for the Einstein tensor of a Levi-Civita connection. A diffeomorphism Noether identity is an off-shell relation among the Euler derivatives of a generally covariant action. A matter Ward identity relates the divergence of the stress tensor to matter Euler derivatives, background sources, anomalies, and boundary terms. The three agree in a restricted Einstein-matter example, but they are not interchangeable.

The distinction matters before imposing the metric equation. If $\mathcal{E}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu}$, then $\nabla^{\mu}\mathcal{E}_{\mu\nu} = 0$ follows geometrically. It does not follow that an arbitrarily prescribed $T_{\mu\nu}$ is conserved. Rather, the equation $\mathcal{E}_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}$ has no solution unless its source obeys the necessary compatibility condition. In an action theory that condition comes from the matter Ward identity and the matter equations. If an external source injects momentum, if flux crosses a boundary, or if the quantum effective action is anomalous, the usual zero-divergence statement is modified.

The same care is required for a gravitational action with fields besides the metric. In a scalar-tensor theory, for example, the divergence of the metric Euler derivative is proportional to the scalar Euler derivative. Calling the metric identity Bianchi and setting it to zero off shell would discard a term required by diffeomorphism invariance. The special pure metric case is important, but it is special.

This paper has four aims. First, it defines a local defect from a specified effective action and derives the full off-shell Noether identity with all normalizations fixed. Second, it separates that identity into gravitational and matter statements and records the hypotheses needed for conservation. Third, it proves covariance under pullback and explains the changes introduced by boundaries and anomalies. Fourth, it shows why a nonzero defect is not a microscopic force law.

The last point is easy to overstate. An Euler derivative says which first variation of an action is nonzero. It supplies neither a clock nor a metric on configuration space. Even in finite dimensions, a potential $V(q)$ is compatible with every gradient flow $\dot{q} = -M(q)\nabla V(q)$ for positive M, with Hamiltonian dynamics after new momentum variables are introduced, and with stochastic dynamics after a noise model is chosen. In a reconstruction problem there is an additional ambiguity: the defect lives in metric field space, while microscopic dynamics lives in the domain of the reconstruction map. Pulling the defect back requires an adjoint and lifting it requires choices that are not contained in $\mathcal{C}_{\mu\nu}$.

Noether's two theorems provide the historical and mathematical basis for the off-shell identities used here [1]. The covariant phase-space distinction between Euler equations, symplectic potential, Noether current, and Noether charge follows Lee and Wald and Iyer and Wald [2, 4]. Boundary generators and surface stress tensors require the separate analyses of Regge and Teitelboim and Brown and York [8, 9]. Local gravitational anomalies and the distinction between consistent and covariant forms are represented by Alvarez-Gaumé and Witten and Bardeen and Zumino [11, 12]. Our purpose is to place these results at the reconstruction interface with the normalizations and nonclaims made explicit.

2 Effective action and reconstruction data

2.1 Reconstructed fields

Let M be a smooth, oriented, time-oriented d-manifold with $d \geq 2$. The Lorentzian metric has signature $(-, +, \dots, +)$. A reconstruction at scale μ is written

$$\mathfrak{R}_\mu : (\mathcal{A}, \rho, \mathcal{G}) \longmapsto (M, g, \psi, \phi; J, \boldsymbol{c}(\mu)). \tag{3}$$

Here $(\mathcal{A}, \rho, \mathcal{G})$ denotes microscopic algebraic, state, and coarse graining data. The symbol ψ collects dynamical fields assigned to the gravitational sector, other than g. The symbol ϕ collects dynamical matter fields. The J's are nondynamical background sources, and $\boldsymbol{c}(\mu)$ are Wilson coefficients.

The split between ψ and ϕ is part of the effective description. It can change under field redefinitions and finite counterterms. Observable statements will therefore be attached either to the total action or to a specified split. A bare symbol $T_{\mu\nu}$, without the action and convention that define it, is insufficient to define a reconstruction defect.

Assumption 2.1 (Well-posed first variation)**.** The reconstructed fields lie in a domain on which the infrared action has a first variation. Boundary conditions and boundary terms are chosen so that the admitted variations define functional derivatives. In the derivation of bulk identities, the generating vector field is initially taken to have compact support in the interior.

This assumption does not say that every microscopic perturbation generates an arbitrary metric perturbation. Completeness of reconstruction variations is a separate question. It is needed to infer the full metric equation from microscopic stationarity, but it is not needed to derive the Ward identity of an already specified infrared action.

2.2 Action and normalization

Write the effective action as

$$\Gamma_{\mathrm{IR}}[g,\psi,\phi;J]=\Gamma_{\mathrm{g}}[g,\psi]+\Gamma_{\mathrm{m}}[g,\psi,\phi;J]. \tag{4}$$

Dependence of the matter action on ψ is allowed when, for example, a dilaton or another gravitational field sets a matter coupling. To avoid duplicating indices, all field components and their tensor labels are included in collective indices a, A, I.

The bulk part of the first variation is normalized as

$$\delta\Gamma_{\mathrm{g}}=\frac{1}{16\pi G_{\mathrm{N}}}\int_M \mathrm{d}^d x\,\sqrt{-g}\,\left(\mathcal{E}_{\mu\nu}\,\delta g^{\mu\nu}+2\mathcal{F}_a\,\delta\psi^a\right)+\int_{\partial M}\boldsymbol{\theta}_{\mathrm{g}}, \tag{5}$$

$$\delta\Gamma_{\mathrm{m}}=\int_M \mathrm{d}^d x\,\sqrt{-g}\,\left(-\frac{1}{2}T_{\mu\nu}\,\delta g^{\mu\nu}+\mathcal{E}^{\mathrm{m}}_A\,\delta\phi^A+\mathcal{O}_I\,\delta J^I+\mathcal{E}^{\mathrm{m},\psi}_a\,\delta\psi^a\right)+\int_{\partial M}\boldsymbol{\theta}_{\mathrm{m}}. \tag{6}$$

Thus

$$T_{\mu\nu}:=-\frac{2}{\sqrt{-g}}\frac{\delta\Gamma_{\mathrm{m}}}{\delta g^{\mu\nu}},\qquad \mathcal{E}_{\mu\nu}:=\frac{16\pi G_{\mathrm{N}}}{\sqrt{-g}}\frac{\delta\Gamma_{\mathrm{g}}}{\delta g^{\mu\nu}}. \tag{7}$$

The sources J^I are transformed when deriving a spurionic Ward identity but are not varied when solving the dynamical equations. Consequently $\mathcal{O}_I=0$ is not imposed unless J^I is promoted to a dynamical field.

The ψ equation of the total action is

$$\frac{1}{8\pi G_{\mathrm{N}}}\mathcal{F}_a+\mathcal{E}^{\mathrm{m},\psi}_a=0. \tag{8}$$

The matter equations are $\mathcal{E}^{\mathrm{m}}_A=0$. The metric equation is

$$\mathcal{C}_{\mu\nu}=0,\qquad \mathcal{C}_{\mu\nu}:=\mathcal{E}_{\mu\nu}-8\pi G_{\mathrm{N}}T_{\mu\nu}. \tag{9}$$

Definition 2.2 (Local reconstruction defect)**.** For a specified local effective action and field split, the local reconstruction defect is the symmetric tensor in equation (9), evaluated on fields returned by $\mathfrak{R}_\mu$. Local means that its value at x is a function of a finite jet of the fields at x, order by order in the derivative expansion.

The definition is deliberately tied to an action. Any symmetric tensor that vanishes on a desired family of metrics is not thereby a reconstruction defect. Multiplying an equation by an invertible differential operator leaves its zero set unchanged but changes its off-shell residual, its differential order, and often its locality. The action normalization in equation (7) removes this ambiguity for the theory under study.

2.3 Local and nonlocal parts

A quantum effective action generally contains both local and nonlocal terms:

$$\Gamma_{\mathrm{IR}}=\Gamma_{\mathrm{loc}}+\Gamma_{\mathrm{nl}}. \tag{10}$$

The Wilsonian local part has a derivative expansion, whereas massless propagation can generate nonanalytic form factors such as $R\log(-\Box/\mu^2)R$ [14, 15]. Accordingly one may write

$$\mathcal{C}_{\mu\nu}=\mathcal{C}^{\mathrm{loc}}_{\mu\nu}+\mathcal{C}^{\mathrm{nl}}_{\mu\nu}. \tag{11}$$

Only the first term is a local defect in the sense of theorem 2.2. Diffeomorphism invariance may constrain the full sum even when neither term is separately invariant under a chosen, scale-dependent split. We will state conservation for whichever functional is itself diffeomorphism invariant.

3 Infinitesimal diffeomorphisms and formal adjoints

Let ξ^μ be a smooth vector field with compact support in the interior. Our active infinitesimal diffeomorphism convention is

$$\delta_\xi g^{\mu\nu} = \mathcal{L}_\xi g^{\mu\nu} = -2\nabla^{(\mu}\xi^{\nu)}. \tag{12}$$

Every other field transforms by its Lie derivative. A scalar s, a covector v_μ, and a contravariant vector w^μ, for example, obey

$$\mathcal{L}_\xi s = \xi^\rho \nabla_\rho s, \tag{13}$$

$$\mathcal{L}_\xi v_\mu = \xi^\rho \nabla_\rho v_\mu + v_\rho \nabla_\mu \xi^\rho, \tag{14}$$

$$\mathcal{L}_\xi w^\mu = \xi^\rho \nabla_\rho w^\mu - w^\rho \nabla_\rho \xi^\mu. \tag{15}$$

Tensor Lie derivatives contain both ξ and $\nabla\xi$. Therefore the scalar formula $\mathcal{E}_A \nabla_\nu \phi^A$ is not the complete Ward operator for arbitrary tensor or spinor matter.

Definition 3.1 (Ward formal adjoint)**.** For a collection of fields χ and dual Euler expressions P, define $\mathcal{W}_\nu(P;\chi)$ by

$$\int_M \sqrt{-g}\, P \cdot \mathcal{L}_\xi \chi = \int_M \sqrt{-g}\, \xi^\nu \mathcal{W}_\nu(P;\chi) \tag{16}$$

for every compactly supported ξ, after integrating derivatives of ξ by parts. Equality is in the distributional sense if the fields are not smooth.

The dot includes all tensor contractions and statistics-dependent conventions. For scalar multiplets,

$$\mathcal{W}_\nu(P;s) = P_A \nabla_\nu s^A. \tag{17}$$

For a covector v_μ,

$$\begin{aligned} \int \sqrt{-g}\, P^\mu \mathcal{L}_\xi v_\mu &= \int \sqrt{-g}\, [P^\mu \xi^\rho \nabla_\rho v_\mu + P^\mu v_\rho \nabla_\mu \xi^\rho] \\ &= \int \sqrt{-g}\, \xi^\nu \left[P^\mu \nabla_\nu v_\mu - \nabla_\mu (P^\mu v_\nu)\right], \end{aligned} \tag{18}$$

so

$$\mathcal{W}_\nu(P;v) = P^\mu \nabla_\nu v_\mu - \nabla_\mu (P^\mu v_\nu). \tag{19}$$

This example shows why the formal-adjoint notation is useful.

For spinors, a Lie derivative requires a lift to the spin bundle, commonly the Kosmann derivative. The resulting identity is most transparent in a vielbein formulation, where local Lorentz and diffeomorphism Ward identities are kept separate. Nothing below assumes that a coordinate-index scalar formula applies to spinors.

Lemma 3.2 (Metric contribution)**.** *For a compactly supported ξ and a symmetric tensor $X_{\mu\nu}$,*

$$\int_M \sqrt{-g}\, X_{\mu\nu} \delta_\xi g^{\mu\nu} = 2 \int_M \sqrt{-g}\, \xi^\nu \nabla^\mu X_{\mu\nu}. \tag{20}$$

Proof. Symmetry gives $X_{\mu\nu}\delta_\xi g^{\mu\nu} = -2X_{\mu\nu}\nabla^\mu \xi^\nu$. Integration by parts produces $2\xi^\nu \nabla^\mu X_{\mu\nu}$. The compact support of ξ removes the boundary integral. □

The compact-support condition is a proof device, not a claim that physically interesting diffeomorphisms vanish at all boundaries. Once ξ reaches a boundary, the discarded term becomes part of a flux or charge statement, as discussed in section 8.

4 The gravitational Noether identity

4.1 General gravitational fields

Assume first that $\Gamma_{\mathrm{g}}[g, \psi]$ is invariant under compactly supported diffeomorphisms and that no gravitational diffeomorphism anomaly is present. Using equations (5) and (20), its variation is

$$\delta_\xi \Gamma_{\mathrm{g}} = \frac{1}{8\pi G_{\mathrm{N}}} \int_M \sqrt{-g}\, \xi^\nu \left[\nabla^\mu \mathcal{E}_{\mu\nu} + \mathcal{W}_\nu(\mathcal{F}; \psi) \right]. \tag{21}$$

We have used the normalization in which the ψ term in equation (5) carries a factor of two.

Theorem 4.1 (Off-shell gravitational Noether identity)**.** *If the gravitational action is invariant under compactly supported diffeomorphisms, then its Euler derivatives satisfy*

$$\boxed{\nabla^\mu \mathcal{E}_{\mu\nu} + \mathcal{W}_\nu(\mathcal{F}; \psi) = 0} \tag{22}$$

off shell.

Proof. Diffeomorphism invariance sets $\delta_\xi \Gamma_{\mathrm{g}} = 0$. Equation (21) then pairs an arbitrary compactly supported ξ^ν with the expression in brackets. The fundamental lemma of the calculus of variations gives equation (22). □

The phrase off shell means that neither the metric equation nor the ψ equation has been imposed. It does not mean that each term in the identity vanishes separately.

Corollary 4.2 (Pure metric action)**.** *If Γ_{g} depends dynamically only on the metric, then*

$$\nabla^\mu \mathcal{E}_{\mu\nu} = 0 \tag{23}$$

off shell.

Proof. There is no ψ Euler derivative, so the second term in equation (22) is absent. □

This result applies to any natural pure metric action for which the functional derivative exists, including higher-curvature actions. For the Einstein-Hilbert action,

$$\Gamma_{\mathrm{g}}^{\mathrm{EH}} = \frac{1}{16\pi G_{\mathrm{N}}} \int_M \sqrt{-g}\, (R - 2\Lambda) + \Gamma_{\partial M}, \tag{24}$$

the Euler tensor is

$$\mathcal{E}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{25}$$

Then equation (23) reduces to the contracted Bianchi identity together with metric compatibility. For a general higher-curvature functional, the same divergence identity is a Noether identity even though the Euler tensor is not the Einstein tensor. Referring to every such identity as the Bianchi identity obscures its variational origin.

4.2 Extra gravitational fields

Consider a scalar ψ in the gravitational sector. Since it transforms as a scalar,

$$\mathcal{W}_\nu(\mathcal{F};\psi) = \mathcal{F}\nabla_\nu\psi, \tag{26}$$

and the identity becomes

$$\nabla^\mu\mathcal{E}_{\mu\nu} = -\mathcal{F}\nabla_\nu\psi. \tag{27}$$

The divergence of the metric Euler derivative therefore vanishes when the gravitational scalar equation $\mathcal{F} = 0$ holds, or in a configuration with $\nabla\psi = 0$, but not as a pure metric identity in general.

Example 4.3 (Nonminimally coupled scalar)**.** Let

$$\Gamma_{\mathrm{g}}[g,\psi] = \frac{1}{16\pi G_{\mathrm{N}}}\int\sqrt{-g}\left[f(\psi)R - \frac{1}{2}(\nabla\psi)^2 - U(\psi)\right]. \tag{28}$$

Direct variation produces metric and scalar Euler derivatives that obey equation (27). Terms involving $\nabla_\mu\nabla_\nu f - g_{\mu\nu}\Box f$ in the metric equation are essential for this cancellation. Dropping the scalar Euler term while keeping an off-shell, spacetime-dependent ψ breaks the identity.

The example also illustrates why sector assignments matter. One may instead call ψ matter and include its stress tensor on the right-hand side. The split identities change, but the total diffeomorphism identity does not, provided the action is unchanged.

5 The matter Ward identity

5.1 Dynamical matter and background sources

Apply a compactly supported diffeomorphism to equation (6). The metric term gives $-\nabla^\mu T_{\mu\nu}$ after integration by parts, while the remaining terms give Ward formal adjoints. It is useful to combine the ψ dependence in the matter action with the other nonmetric matter terms:

$$\mathcal{W}^{\mathrm{m}}_\nu := \mathcal{W}_\nu(\mathcal{E}^{\mathrm{m}};\phi) + \mathcal{W}_\nu(\mathcal{E}^{\mathrm{m},\psi};\psi), \qquad \mathcal{W}^J_\nu := \mathcal{W}_\nu(\mathcal{O};J). \tag{29}$$

Theorem 5.1 (Off-shell matter Ward identity)**.** *If $\Gamma_{\mathrm{m}}[g,\psi,\phi;J]$ is invariant when all dynamical fields and background sources are transformed, then*

$$\boxed{\nabla^\mu T_{\mu\nu} = \mathcal{W}^{\mathrm{m}}_\nu + \mathcal{W}^J_\nu} \tag{30}$$

in the interior.

Proof. Using the first variation gives

$$0 = \delta_\xi\Gamma_{\mathrm{m}} = \int_M \sqrt{-g}\,\xi^\nu\left[-\nabla^\mu T_{\mu\nu} + \mathcal{W}^{\mathrm{m}}_\nu + \mathcal{W}^J_\nu\right]. \tag{31}$$

Arbitrariness of the compactly supported vector field gives equation (30). □

The theorem is off shell because it retains the matter Euler derivatives. It is a spurionic identity in the sources: the J^I's are transformed to derive the equation, even though they remain fixed when the dynamical equations are solved.

Corollary 5.2 (On-shell matter conservation)**.** *Suppose that the dynamical matter equations hold, the ψ dependence of Γ_{m} is either absent or included in a satisfied total ψ equation, and the source Ward term vanishes. Then*

$$\nabla^\mu T_{\mu\nu} = 0. \tag{32}$$

The source Ward term can vanish because sources are absent, are covariantly constant in the relevant representation, or have a configuration whose formal adjoint vanishes. The fact that a source is nondynamical does not by itself make the term vanish.

5.2 Scalar example

For scalar matter ϕ^A and scalar sources J^I, equation (30) reads

$$\nabla^\mu T_{\mu\nu} = \mathcal{E}^{\mathrm{m}}_A \nabla_\nu \phi^A + \mathcal{E}^{\mathrm{m},\psi}_a \nabla_\nu \psi^a + \mathcal{O}_I \nabla_\nu J^I. \tag{33}$$

On the ϕ equations, and in the absence of ψ couplings,

$$\nabla^\mu T_{\mu\nu} = \mathcal{O}_I \nabla_\nu J^I. \tag{34}$$

The right side is a force density or momentum transfer from the prescribed background.

Example 5.3 (Driven scalar)**.** Consider

$$\Gamma_{\mathrm{m}}[g, \phi; J] = -\int_M \sqrt{-g}\left[\frac{1}{2}\nabla_\mu \phi \nabla^\mu \phi + V(\phi) + J(x)\phi\right]. \tag{35}$$

With the sign conventions above, $\mathcal{O} = -\phi$. On the ϕ equation,

$$\nabla^\mu T_{\mu\nu} = -\phi \nabla_\nu J. \tag{36}$$

A spatially or temporally varying source exchanges momentum with the scalar system. The total stress tensor of a larger dynamical apparatus that generates J may be conserved, but the stress tensor of the driven subsystem is not.

5.3 Gauge fields and spin

For fields carrying spacetime indices, the Ward operator contains derivatives of the Euler expressions, as in equation (19). Gauge redundancy supplies additional Noether identities. For a Yang-Mills field, one should not infer the diffeomorphism Ward identity by treating every component as a scalar. A gauge-covariant decomposition combines a field-strength force term with the gauge equation; on the full matter and gauge equations the ordinary source-free stress tensor is conserved.

Spinor matter is most naturally varied with respect to a vielbein. The antisymmetric part of the vielbein equation is related to the local Lorentz Ward identity, while the diffeomorphism identity controls the covariant divergence. Improvement terms can move derivatives of spin currents between canonical and Hilbert stress tensors. The stress tensor defined by equation (7) is the metric or Hilbert tensor for the chosen effective action, so the relevant identity is the one obtained from that variation.

6 The defect identity and conservation compatibility

6.1 Combined off-shell identity

The gravitational and matter identities can be combined without imposing either set of equations. From $\mathcal{C}_{\mu\nu} = \mathcal{E}_{\mu\nu} - 8\pi G_{\mathrm{N}} T_{\mu\nu}$,

$$\nabla^\mu \mathcal{C}_{\mu\nu} = \nabla^\mu \mathcal{E}_{\mu\nu} - 8\pi G_{\mathrm{N}} \nabla^\mu T_{\mu\nu}. \tag{37}$$

Using equations (22) and (30) gives the main identity.

Theorem 6.1 (Off-shell reconstruction-defect identity)**.** *For an anomaly-free diffeomorphism-invariant action, with compactly supported diffeomorphisms and all sources transformed, the local defect satisfies*

$$\boxed{\nabla^\mu \mathcal{C}_{\mu\nu} = -\mathcal{W}_\nu(\mathcal{F};\psi) - 8\pi G_{\mathrm{N}} \left(\mathcal{W}^{\mathrm{m}}_\nu + \mathcal{W}^{J}_\nu\right).} \tag{38}$$

Proof. Substitute equation (22) for the first term in equation (37) and equation (30) for the second. □

If ψ occurs in both sector actions, it is sometimes more useful to combine its terms using the total equation equation (8). For scalar ψ^a, the combined contribution is

$$-\left[\mathcal{F}_a + 8\pi G_{\mathrm{N}} \mathcal{E}^{\mathrm{m},\psi}_a\right] \nabla_\nu \psi^a. \tag{39}$$

It vanishes on the total ψ equation. Analogous statements hold for tensor fields with their formal adjoints.

Corollary 6.2 (Conservation compatibility)**.** *Assume:*

(i) the gravitational action is diffeomorphism invariant and all additional gravitational field equations hold;

(ii) the dynamical matter equations hold;

(iii) background sources contribute no Ward force;

(iv) there is no diffeomorphism anomaly; and

(v) the bulk identity is tested with compact support, or all relevant boundary flux terms vanish.

Then

$$\nabla^\mu \mathcal{C}_{\mu\nu} = 0. \tag{40}$$

Proof. Under the stated assumptions every term on the right side of equation (38) vanishes. Boundary and anomaly qualifications are made explicit in sections 8 and 9. □

This corollary is a compatibility statement. It says that the residual lies in the divergence-free subspace selected by the gauge identity. It does not say that the residual vanishes.

6.2 Conservation does not imply the metric equation

Proposition 6.3 (Nonzero conserved defect)**.** *On any metric-compatible spacetime, for every nonzero constant* λ*,*

$$\mathcal{C}_{\mu\nu} = \lambda g_{\mu\nu} \tag{41}$$

is nonzero and covariantly conserved.

Proof. Metric compatibility gives $\nabla^\mu g_{\mu\nu} = 0$. Since λ is constant, $\nabla^\mu(\lambda g_{\mu\nu}) = 0$, while nondegeneracy of g and $\lambda \neq 0$ imply $\mathcal{C} \neq 0$. □

The example can be realized as a mismatch in the cosmological coupling. If the action used by the reconstruction has cosmological constant Λ but the reconstructed metric solves an equation with $\Lambda - \lambda$, the defect contains $\lambda g_{\mu\nu}$. No Ward identity selects $\lambda = 0$.

More generally, the Euler derivative of any diffeomorphism-invariant pure metric functional is divergence-free. Thus the kernel of the divergence operator is large. Gauge compatibility removes longitudinal inconsistency; it does not solve the transverse field equation.

Remark 6.4 (Equation versus identity)*.* The metric equation $\mathcal{C}_{\mu\nu} = 0$ restricts admissible configurations. The Noether identity $\nabla^\mu \mathcal{C}_{\mu\nu} = 0$, when its auxiliary equations and caveats hold, is a differential dependence among the equations. It reflects gauge redundancy. Treating the identity as an equation that dynamically drives $\mathcal{C}$ to zero reverses their logical roles.

6.3 Bianchi is not a matter equation

Suppose one writes

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_{\mathrm{N}} S_{\mu\nu} \tag{42}$$

for a prescribed symmetric tensor $S_{\mu\nu}$. Taking a divergence gives the necessary condition

$$\nabla^\mu S_{\mu\nu} = 0. \tag{43}$$

This calculation does not prove equation (43) for an arbitrary S. It proves that no solution of equation (42) can exist where the condition fails. If $S = T$ comes from a diffeomorphism-invariant matter action, the matter Ward identity explains when the condition is satisfied. If S is phenomenological input, its conservation must be checked or supplied independently.

7 Pullback covariance

7.1 Naturality of the effective action

Let $f : M \to M$ be an orientation- and time-orientation-preserving diffeomorphism. A natural action satisfies

$$\Gamma_{\mathrm{IR}}[f^*g, f^*\psi, f^*\phi; f^*J] = \Gamma_{\mathrm{IR}}[g, \psi, \phi; J], \tag{44}$$

with transformed boundary data when a boundary is present. Naturality is stronger than coordinate covariance of a displayed formula. It excludes untransformed preferred coordinate functions and fixed structures that have not been included among the sources.

Theorem 7.1 (Pullback covariance of Euler derivatives)**.** *Let an action satisfy equation (44), and suppose its first variation is unique on the admitted field domain. Then its normalized Euler derivatives transform in their natural tensor representations. In particular,*

$$\mathcal{E}[f^*g, f^*\psi] = f^*\mathcal{E}[g, \psi], \tag{45}$$

$$T[f^*g, f^*\psi, f^*\phi; f^*J] = f^*T[g, \psi, \phi; J]. \tag{46}$$

Proof. Choose an arbitrary compactly supported metric variation $h^{\mu\nu}$ at the unpulled configuration. The corresponding variation at the pulled-back configuration is f^*h. Differentiate

$$\Gamma_{\mathrm{IR}}[f^*(g+\epsilon h), f^*\psi, f^*\phi; f^*J] = \Gamma_{\mathrm{IR}}[g+\epsilon h, \psi, \phi; J] \tag{47}$$

at $\epsilon = 0$. Express both sides by their first variations. A change of variables converts the pulled-back volume form and contractions into the unpulled integral. Because h is arbitrary and the functional derivative is unique, the metric Euler densities agree after pullback. Division by the natural volume density gives equations (45) and (46). The argument for the other fields is identical in their respective bundles. □

Corollary 7.2 (Pullback covariance of the defect)**.** *Under the hypotheses of theorem 7.1,*

$$\boxed{\mathcal{C}[f^*g, f^*\psi, f^*\phi; f^*J] = f^*\mathcal{C}[g, \psi, \phi; J].} \tag{48}$$

Proof. Use the linear definition $\mathcal{C} = \mathcal{E} - 8\pi G_{\mathrm{N}} T$, regard G_{N} as a scalar coupling, and apply equations (45) and (46). □

7.2 Covariance of the divergence identity

The Levi-Civita connection is natural:

$$\nabla^{f^*g}(f^*X) = f^*(\nabla^g X) \tag{49}$$

for every tensor X. Consequently

$$\nabla^{\mu}_{f^*g}\mathcal{C}_{\mu\nu}[f^*\text{fields}] = f^*\left(\nabla^{\mu}_{g}\mathcal{C}_{\mu\nu}[\text{fields}]\right). \tag{50}$$

If a configuration satisfies the conservation compatibility condition, every diffeomorphic representative does as well.

This statement concerns simultaneous pullback of all data. Holding a background source fixed while pulling back only the metric is not a gauge transformation of the sourced theory. The difference appears precisely in the source Ward term. Likewise, pulling back a manifold with boundary while leaving the boundary embedding or boundary conditions fixed need not preserve the variational problem.

7.3 Covariant reconstruction dictionaries

Let microscopic descriptions carry an action of a group of relabelings α_f corresponding to spacetime diffeomorphisms. A covariant reconstruction dictionary obeys

$$\mathfrak{R}_\mu(\alpha_f z) = f^*\mathfrak{R}_\mu(z), \tag{51}$$

where $z = (\mathcal{A}, \rho, \mathcal{G})$. Combining equations (48) and (51) gives

$$\mathcal{C}[\mathfrak{R}_\mu(\alpha_f z)] = f^*\mathcal{C}[\mathfrak{R}_\mu(z)]. \tag{52}$$

Proposition 7.3 (Gauge-orbit consistency)**.** *If the action and reconstruction map are both natural, then defect vanishing and defect conservation are constant properties along a diffeomorphism orbit.*

Proof. Pullback by a diffeomorphism is invertible. Therefore $f^*\mathcal{C} = 0$ if and only if $\mathcal{C} = 0$. Equation (50) gives the same equivalence for the divergence. □

A reconstruction tied to a preferred coordinate chart can fail equation (51). Its output may still be written with tensor indices, but the dictionary has introduced untracked background structure. Pullback covariance must then be restored by transforming that structure as a source, or else it is absent.

8 Boundaries, currents, and charges

8.1 The bulk proof and its surface term

Without compact support, theorem 3.2 reads

$$\int_M \sqrt{-g}\, X_{\mu\nu}\delta_\xi g^{\mu\nu} = 2\int_M \sqrt{-g}\, \xi^\nu \nabla^\mu X_{\mu\nu} - 2\int_{\partial M} \sqrt{|h|}\, n^\mu X_{\mu\nu}\xi^\nu, \tag{53}$$

with the standard modification for null boundaries. Variations of other fields add their own surface terms. A bulk Ward identity follows locally from vector fields supported away from the boundary. A global conservation law requires the boundary terms to be included.

For the Einstein-Hilbert action on a smooth non-null boundary with Dirichlet metric data, the Gibbons-Hawking-York term makes the variational problem well-posed [6, 7]. Null segments, joints, corners, higher-curvature actions, and mixed boundary conditions require other completions. One cannot infer a boundary charge from a bulk Lagrangian while ignoring the terms that define its first variation.

8.2 Covariant phase space

In differential-form notation, let the Lagrangian be a d-form $\boldsymbol{L}$. Its variation has the form

$$\delta\boldsymbol{L} = \boldsymbol{E}(\chi)\cdot\delta\chi + \mathrm{d}\boldsymbol{\theta}(\chi,\delta\chi), \tag{54}$$

where χ denotes all fields and $\boldsymbol{\theta}$ is a symplectic-potential $(d-1)$-form. The Noether current associated with ξ is

$$\boldsymbol{J}_\xi := \boldsymbol{\theta}(\chi, \mathcal{L}_\xi\chi) - \iota_\xi \boldsymbol{L}. \tag{55}$$

For a diffeomorphism-covariant Lagrangian,

$$\mathrm{d}\boldsymbol{J}_\xi = -\boldsymbol{E}(\chi)\cdot\mathcal{L}_\xi\chi. \tag{56}$$

Thus the current is closed on shell. Locally, after separating constraint terms, it can be written as a Noether charge form plus terms proportional to the equations of motion [2, 4].

Three statements must be kept distinct:

(a) the off-shell Noether identity among Euler derivatives;

(b) on-shell closure of the Noether current; and

(c) integrability and conservation of a surface charge on a chosen phase space with boundary conditions.

The first is used in theorem 6.1. The second follows from equation (56). The third may fail because of symplectic flux, nonintegrability, or an inadmissible generator.

Regge and Teitelboim showed in Hamiltonian language that differentiable generators require surface terms and that those terms carry the physical charges [8]. Brown and York defined a quasilocal surface stress tensor by varying the on-shell action with respect to boundary metric data [9]. Wald and Zoupas treated charges in settings with radiation through null infinity, where flux is part of the balance law [5]. These constructions agree in overlapping regimes only after their boundary conditions and reference terms are matched.

8.3 Boundary balance law

Let $U \subset M$ be a spacetime region. Integrating the matter Ward identity against a vector field gives schematically

$$\int_{\partial U} \mathrm{d}\Sigma_\mu \, T^\mu{}_\nu \xi^\nu = \int_U \sqrt{-g}\, \left[T^\mu{}_\nu \nabla_\mu \xi^\nu + \xi^\nu (\mathcal{W}^{\mathrm{m}}_\nu + \mathcal{W}^J_\nu) \right]. \tag{57}$$

For a Killing field, on the matter equations and without source forces, the bulk terms vanish and the boundary fluxes balance. If part of ∂U is an open timelike or null boundary, the corresponding flux need not be zero. Calling the interior subsystem conserved or nonconserved without specifying the boundary is incomplete.

Distributional sources introduce a related issue. A thin shell can carry a surface stress tensor whose tangential divergence is tied to the jump in normal bulk flux. The correct Ward identity then contains both bulk and surface distributions. Deleting the surface term can make a globally conserved system appear to violate the identity.

9 Diffeomorphism anomalies

9.1 Anomalous variation

At the quantum level the measure and regulator may fail to preserve diffeomorphism invariance. Fix the sign convention

$$\delta_\xi \Gamma = -\int_M \sqrt{-g}\, \xi^\nu \mathcal{A}_\nu \tag{58}$$

for a local consistent anomaly $\mathcal{A}_\nu$. Repeating the matter derivation gives

$$\nabla^\mu T_{\mu\nu} = \mathcal{W}^{\mathrm{m}}_\nu + \mathcal{W}^J_\nu + \mathcal{A}_\nu. \tag{59}$$

With the same convention for the total action,

$$\nabla^\mu \mathcal{C}_{\mu\nu} = -\mathcal{W}_\nu(\mathcal{F};\psi) - 8\pi G_{\mathrm{N}} \left(\mathcal{W}^{\mathrm{m}}_\nu + \mathcal{W}^J_\nu + \mathcal{A}_\nu \right). \tag{60}$$

The sign of a quantity called the anomaly varies in the literature; equation (58) fixes it here.

A local gravitational anomaly is possible for chiral matter in appropriate dimensions. Its cancellation is a consistency condition for treating diffeomorphisms as an exact gauge redundancy [11]. Anomaly inflow from a bulk theory can cancel a boundary anomaly, in which case the combined bulk-boundary system has the appropriate Ward identity even though either piece does not.

9.2 Consistent and covariant forms

The anomaly obtained by varying the effective action is the consistent anomaly and satisfies the Wess-Zumino consistency condition [10]. Adding a Bardeen-Zumino local polynomial produces a covariant current or stress tensor with a covariant anomaly [12]. The two objects transform differently and obey different displayed Ward equations, although they encode the same underlying obstruction once the local polynomial is tracked.

Therefore an anomaly statement must specify:

(i) whether the effective action or a covariantized current is being varied;

(ii) the finite local counterterm scheme;

(iii) whether bulk inflow and boundary degrees of freedom are included; and

(iv) whether the anomaly is local or global.

The local infinitesimal identity above does not diagnose global anomalies under large diffeomorphisms.

Proposition 9.1 (Anomaly obstruction). *If $\mathcal{A}_\nu \neq 0$ and the other Euler and source terms vanish, the ordinary defect conservation equation fails:*

$$\nabla^\mu \mathcal{C}_{\mu\nu} = -8\pi G_{\mathrm{N}} \mathcal{A}_\nu. \tag{61}$$

Proof. Set the other terms in equation (60) to zero. □

This failure cannot be repaired by invoking $\nabla^\mu G_{\mu\nu} = 0$. If the metric equation equates a divergence-free gravitational tensor to an anomalous matter stress tensor without inflow or additional degrees of freedom, the equations are inconsistent.

10 Locality, splitting, and scheme dependence

10.1 Which defect is local?

The definition in theorem 2.2 is appropriate for a Wilsonian action truncated at a specified derivative order. A one-particle-irreducible effective action after integrating out massless fields is generally nonlocal. Its functional derivative is still a valid Euler residual, and its total diffeomorphism identity still holds, but evaluating it at x can depend on fields away from x and on a Green-function prescription.

For example,

$$\Gamma_{\mathrm{nl}} \supset \int_M \sqrt{-g}\, R \log(-\Box/\mu^2) R \tag{62}$$

is covariant after a prescription for the operator logarithm is supplied. Its metric derivative contains nonlocal kernels. Calling that contribution a local mismatch at x would be false. One may instead report the pair $(\mathcal{C}^{\mathrm{loc}}, \mathcal{C}^{\mathrm{nl}})$, with the split and scale stated.

10.2 Counterterms and sector shifts

Let $\Delta\Gamma[g, \psi]$ be a finite local diffeomorphism-invariant functional. Moving it from the matter sector to the gravitational sector,

$$\Gamma'_{\mathrm{g}} = \Gamma_{\mathrm{g}} + \Delta\Gamma, \qquad \Gamma'_{\mathrm{m}} = \Gamma_{\mathrm{m}} - \Delta\Gamma, \tag{63}$$

changes $\mathcal{E}_{\mu\nu}$ and $T_{\mu\nu}$ separately. The total metric coefficient does not change:

$$\mathcal{E}'_{\mu\nu} - 8\pi G_{\mathrm{N}} T'_{\mu\nu} = \mathcal{E}_{\mu\nu} - 8\pi G_{\mathrm{N}} T_{\mu\nu}. \tag{64}$$

This is immediate from varying $\Gamma'_{\mathrm{g}} + \Gamma'_{\mathrm{m}} = \Gamma_{\mathrm{g}} + \Gamma_{\mathrm{m}}$. The split identities are scheme-dependent, while the defect of the fixed total action is not.

Renormalization changes Wilson coefficients and composite-operator definitions with μ. A meaningful comparison of defects at two scales must run the couplings and fields consistently. The dependence $\mathcal{C}_{\mu\nu}(\mu)$ is not by itself a time evolution of the reconstructed metric. Renormalization-group scale and physical time are different parameters.

10.3 Field redefinitions

Under an invertible local field redefinition, Euler derivatives transform by the adjoint Jacobian of the redefinition. Their common zero locus is preserved, but their components and apparent differential order may change. The pullback-covariance theorem concerns diffeomorphisms of spacetime, not arbitrary reparametrizations of field space. A claim about the numerical size of a defect therefore requires a field-space norm and a choice of variables in addition to the action normalization.

This qualification does not make $\mathcal{C}_{\mu\nu}$ meaningless. It fixes what is invariant: whether the specified metric equation holds, how its residual transforms as a tensor in a fixed formulation, and which Ward identity it obeys. It rules out comparing unnormalized residual components from inequivalent formulations as though they were absolute observables.

11 Why the defect is not a relaxation law

11.1 First variation on reconstruction space

Let z denote microscopic or mesoscopic reconstruction data and write

$$(g(z), \psi(z), \phi(z); J(z)) = \mathfrak{R}_\mu(z). \tag{65}$$

For an admissible variation δz, the chain rule gives

$$\begin{aligned} \delta(\Gamma_{\mathrm{IR}} \circ \mathfrak{R}_\mu)_z = {} & \frac{1}{16\pi G_{\mathrm{N}}} \left\langle \mathcal{C}, Dg_z[\delta z] \right\rangle_g \\ & + \left\langle \frac{1}{8\pi G_{\mathrm{N}}} \mathcal{F} + \mathcal{E}^{\mathrm{m},\psi}, D\psi_z[\delta z] \right\rangle \\ & + \left\langle \mathcal{E}^{\mathrm{m}}, D\phi_z[\delta z] \right\rangle + \left\langle \mathcal{O}, DJ_z[\delta z] \right\rangle . \end{aligned} \tag{66}$$

Here the brackets denote the spacetime pairings determined by the first variation and boundary conditions. If the nonmetric equations hold and sources are fixed, the microscopic first variation is

$$\delta(\Gamma_{\mathrm{IR}} \circ \mathfrak{R}_\mu)_z = \frac{1}{16\pi G_{\mathrm{N}}} \left\langle (Dg_z)^* \mathcal{C}, \delta z \right\rangle_z . \tag{67}$$

An adjoint requires pairings on both spaces. The action supplies the covector $\delta(\Gamma_{\mathrm{IR}} \circ \mathfrak{R}_\mu)$, but not a canonical vector field $\dot{z}$.

Proposition 11.1 (Projected stationarity)**.** *Stationarity of the pulled-back action implies*

$$(Dg_z)^* \mathcal{C} = 0 \tag{68}$$

after the other equations hold. It implies $\mathcal{C} = 0$ only if the image of Dg_z is sufficiently complete, modulo gauge and boundary directions, to separate admissible defects.

Proof. Equation (67) vanishes for every admissible δz exactly when the adjoint expression vanishes. If the annihilator of $\operatorname{Ran} Dg_z$ contains nonzero tensors, such a tensor can satisfy equation (68). Triviality of that annihilator on the physical quotient is the required completeness condition. $\square$

This projection issue already prevents a unique inference from defect to microscopic change. Components of $\mathcal{C}$ orthogonal to $\operatorname{Ran} Dg_z$ are invisible to variations of z. Elements of $\operatorname{Ker} Dg_z$ change the microscopic data without changing the reconstructed metric to first order. A lift from metric-space motion to microscopic motion is therefore generally neither existent nor unique.

11.2 Mobility is additional structure

In finite dimensions, let $F_i = \partial_i V$ be a nonzero covector. To turn it into a velocity, choose a positive contravariant tensor M^{ij}:

$$\dot{q}^i = -M^{ij} F_j. \tag{69}$$

Then

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -F_i M^{ij} F_j \leq 0. \tag{70}$$

Every positive M gives a different flow while preserving the same stationary points. The potential does not select M.

For fields, a mobility can be ultralocal, elliptic, hyperbolic after auxiliary variables are introduced, or nonlocal. It can project out gauge directions or couple to them before gauge fixing. Positivity may be appropriate for a Euclidean dissipative flow but not for Lorentzian Hamiltonian evolution. A Lorentzian metric field space also has no automatic positive norm with which to identify the covector $\mathcal{C}$ and a vector.

Theorem 11.2 (Nonuniqueness of defect-driven flows)**.** *Let $F(z)$ be a nonzero Euler covector at a configuration z in a configuration space of dimension at least two. There are infinitely many positive-definite mobilities M for which $\dot{z} = -MF$ decreases the same action, and the resulting velocities are not all equal.*

Proof. Choose a basis in which $F = (f, 0, \ldots, 0)$ with $f \neq 0$. For every $\alpha > 0$, the positive diagonal matrix $M_\alpha = \mathrm{diag}(\alpha, 1, \ldots, 1)$ gives velocity $(-\alpha f, 0, \ldots, 0)$. Distinct values of α give distinct velocities, and $F^T M_\alpha F = \alpha f^2 > 0$. More general positive matrices can also add components transverse to the chosen basis representation of F. □

The one-dimensional case remains nonunique through the arbitrary positive scalar mobility and through time reparametrization. Thus nonuniqueness is not an artifact of a large field space.

11.3 Inequivalent dynamics with the same residual

The same effective action can participate in several types of evolution:

$$\text{gradient:} \quad \dot{z} = -M\,\frac{\delta\Gamma}{\delta z}, \tag{71}$$

$$\text{inertial:} \quad K\,\ddot{z} + D\,\dot{z} = -\frac{\delta\Gamma}{\delta z}, \tag{72}$$

$$\text{Hamiltonian:} \quad \dot{z} = \frac{\delta H}{\delta p}, \qquad \dot{p} = -\frac{\delta H}{\delta z}, \tag{73}$$

$$\text{stochastic:} \quad \mathrm{d}z = -M\,\frac{\delta\Gamma}{\delta z}\,\mathrm{d}t + \sigma\,\mathrm{d}W_t. \tag{74}$$

The operators M, K, D, σ, the momenta p, the Hamiltonian H, and the stochastic convention are additional data. Causality and well-posedness also require choices of initial conditions and Green functions. Agreement on the stationary residual does not imply agreement on transients, fluctuations, or signal propagation.

For a reconstruction map, a proposed microscopic gradient law might be

$$\dot{z} = -M_z (Dg_z)^* \mathcal{C}. \tag{75}$$

Neither M_z nor the pairing used to define $(Dg_z)^*$ is fixed by $\mathcal{C}$. Moreover, any $k \in \mathrm{Ker}\, Dg_z$ can be added to $\dot{z}$ without changing the first-order metric velocity. Conversely, a desired metric velocity need not lie in $\mathrm{Ran}\, Dg_z$.

Corollary 11.3 (No universal microscopic force). *A nonzero reconstruction defect, by itself, determines no unique microscopic relaxation law.*

Proof. The defect is an Euler covector in metric field space. Equations (67) and (75) show that even a gradient interpretation needs an adjoint pairing and a mobility. Theorem 11.2 gives infinitely many choices after those spaces are finite dimensional, while kernel and range ambiguities of Dg_z add further nonuniqueness. Other dynamical classes in equations (72) to (74) are inequivalent. □

12 Counterexamples and failure modes

The hypotheses in the preceding theorems are independent enough that removing one changes the conclusion. This section collects explicit tests that can be applied to a proposed reconstruction.

12.1 A conserved defect need not vanish

The tensor $\lambda g_{\mu\nu}$ in theorem 6.3 is the simplest counterexample to

$$\nabla^\mu \mathcal{C}_{\mu\nu} = 0 \quad \Longrightarrow \quad \mathcal{C}_{\mu\nu} = 0. \tag{76}$$

Transverse gravitational-wave residuals and Euler derivatives of higher-curvature invariants give less trivial examples. Boundary or initial data, not the divergence identity, decide whether these modes vanish.

12.2 A prescribed source can violate compatibility

Take a smooth symmetric tensor $S_{\mu\nu}$ with compact support and $\nabla^\mu S_{\mu\nu} \neq 0$. It cannot appear alone on the right side of the Einstein equation. The contracted Bianchi identity detects the inconsistency; it does not manufacture a matter equation that makes S conserved. An external agency, additional stress tensor, or modified gravitational equation is required.

12.3 A background source carries momentum

The driven scalar in theorem 5.3 satisfies its matter equation but has $\nabla^\mu T_{\mu\nu} = -\phi \nabla_\nu J$. The source-free conclusion fails solely because $J(x)$ varies. Treating J as a scalar spurion makes the action formally covariant and exposes, rather than removes, the force-density term.

12.4 An extra gravitational field is off shell

For the scalar-tensor action in theorem 4.3, $\nabla^\mu \mathcal{E}_{\mu\nu} = -\mathcal{F} \nabla_\nu \psi$. Unless $\mathcal{F} = 0$ or ψ is constant, the metric Euler tensor is not separately divergence-free. The pure metric identity cannot be imported into an enlarged gravitational field space.

12.5 A boundary leaks flux

Let a matter wave packet leave a finite region through a timelike boundary. The local interior identity can hold at every smooth point while the integrated energy inside the region decreases. The decrease equals the boundary flux. Imposing a zero-flux balance law without reflective boundary conditions contradicts the physical configuration.

12.6 A quantum anomaly remains uncanceled

If chiral matter has a local gravitational anomaly and there is no inflow or anomaly-canceling field content, then $\nabla^\mu T_{\mu\nu} = \mathcal{A}_\nu$ on the matter equations. A divergence-free pure metric Euler tensor cannot equal this stress tensor. The failure is a gauge-consistency obstruction, not a small reconstruction error to be relaxed away.

12.7 The effective action is nonlocal

If the relevant action contains equation (62), its defect can depend on the fields throughout a causal or Euclidean domain. A local finite-jet diagnostic omits part of the equation. Diffeomorphism covariance does not imply locality, and Ward compatibility does not repair the omission.

12.8 The reconstruction dictionary is not natural

Suppose a reconstruction selects components by reference to a fixed coordinate function x^0 that is not transformed as a background scalar. Then generally

$$\mathfrak{R}_\mu(\alpha_f z) \neq f^*\mathfrak{R}_\mu(z). \tag{77}$$

The resulting component array may transform incorrectly even if the subsequent action is covariant. Naturality must hold at both the dictionary and action levels for equation (52).

12.9 Stationarity probes only a projection

Let the reconstructed metrics vary only conformally:

$$\delta g^{\mu\nu} = -2\sigma g^{\mu\nu}. \tag{78}$$

Then metric stationarity tests only

$$g^{\mu\nu}\mathcal{C}_{\mu\nu} = 0. \tag{79}$$

A nonzero traceless defect is invisible. No Ward identity enlarges the image of the reconstruction tangent map.

12.10 The defect is mistaken for a beta function

The coefficient μ in $\mathfrak{R}_\mu$ labels coarse-graining scale. A renormalization-group equation can describe how couplings and fields are reparametrized as μ changes. It is not a Lorentzian time-evolution law unless a separate construction identifies the scale flow with physical dynamics. Writing $\partial_{\log\mu} g_{\mu\nu} \propto -\mathcal{C}_{\mu\nu}$ is an additional ansatz, not a consequence of the Ward identity.

12.11 A field-space norm is left unspecified

A statement that the defect is small needs a norm, smearing, or dimensionless observable. Lorentzian contraction $\mathcal{C}_{\mu\nu}\mathcal{C}^{\mu\nu}$ is not positive definite. Pointwise components are coordinate dependent. An operational bound can instead smear the defect against normalized test tensors or compare its terms in a controlled derivative expansion. The Ward identity alone supplies no norm.

13 A hierarchy of statements

It is useful to order the conclusions by the data they require.

Table 1: Assumption removed and the corresponding obstruction.

Missing input	Consequence
Matter equations	$\nabla^\mu T_{\mu\nu}$ contains matter Euler terms.
Source neutrality	External backgrounds inject the force density $\mathcal{W}^J_\nu$.
Extra gravitational equations	$\nabla^\mu \mathcal{E}_{\mu\nu}$ contains $\mathcal{W}_\nu(\mathcal{F};\psi)$.
Anomaly cancellation	The Ward identity contains $\mathcal{A}_\nu$.
Boundary completion	Bulk integration by parts misses flux or charge terms.
Action locality	The defect need not be a finite-jet local tensor.
Natural reconstruction	Pullback covariance can fail before the action is evaluated.
Complete reconstruction tangent	Microscopic stationarity tests only a projection of $\mathcal{C}$.
Mobility and time structure	The defect does not define a microscopic velocity.

Tensor definition. A specified effective action and field split define $\mathcal{C}_{\mu\nu}$ through equation (9). This is an off-shell diagnostic.

Pullback covariance. Naturality of the action and simultaneous transformation of all fields, sources, and boundary data imply $\mathcal{C}[f^*\text{fields}] = f^*\mathcal{C}[\text{fields}]$. Equivariance of the reconstruction dictionary is additionally required to carry that statement back to microscopic relabelings.

Ward identity. Infinitesimal diffeomorphism invariance gives equation (38). This is an identity among Euler expressions, not an equation imposed on initial data.

Conservation compatibility. The nonmetric equations, source neutrality, anomaly cancellation, and appropriate boundary conditions reduce the identity to $\nabla^\mu \mathcal{C}_{\mu\nu} = 0$.

Metric equation. Stationarity under arbitrary physical metric variations gives $\mathcal{C}_{\mu\nu} = 0$. Microscopic stationarity gives the same conclusion only when the reconstruction tangent is complete on the physical quotient.

Microscopic dynamics. A relaxation or evolution law requires further kinetic, symplectic, mobility, causal, stochastic, and lifting data. It is not fixed by any of the previous statements.

Each level can hold while the next fails. A natural defect may not be conserved because matter is driven. A conserved defect may be nonzero. A vanishing defect may characterize equilibrium without determining the path to equilibrium. This hierarchy is the main safeguard against turning a kinematic consistency relation into an unsupported microscopic dynamics.

14 Conclusion

The reconstruction defect is the metric Euler residual of a specified infrared action,

$$\mathcal{C}_{\mu\nu} = \mathcal{E}_{\mu\nu} - 8\pi G_{\mathrm{N}} T_{\mu\nu}.$$

Its normalization follows from the first variation and fixes every factor of $8\pi G_{\mathrm{N}}$. For a pure metric gravitational action, diffeomorphism invariance gives $\nabla^\mu \mathcal{E}_{\mu\nu} = 0$ off shell. The Einstein-Hilbert case realizes this statement through the contracted Bianchi identity, but a general gravitational action realizes a Noether identity. Additional gravitational fields contribute their Euler derivatives and must not be omitted.

The matter statement is different. Diffeomorphism invariance yields a Ward identity in which the divergence of $T_{\mu\nu}$ equals matter Euler terms and background-source forces. Ordinary conservation follows only on the dynamical matter equations and only when sources do not exchange momentum. Boundary flux and diffeomorphism anomalies supply further, separately controlled terms. Under the complete set of hypotheses the defect is covariantly conserved, which makes the metric equation compatible with gauge redundancy.

Naturality of the action makes the defect covariant under simultaneous pullback of every field and source. Naturality of the reconstruction dictionary then carries that covariance to microscopic relabelings. Boundaries require more than a bulk tensor calculation: the symplectic potential, differentiable generators, flux, and surface charges depend on the completed variational problem.

None of these identities drives a nonzero defect to zero. A conserved defect can remain nonzero, as $\lambda g_{\mu\nu}$ shows. At the reconstruction interface, microscopic stationarity tests the adjoint projection $(Dg)^*\mathcal{C}$, and only a complete tangent image recovers the full equation. Turning the resulting covector into microscopic motion requires a clock, pairings, a mobility or symplectic structure, causal data, and a lift through the reconstruction map. Different choices give inequivalent dynamics with the same stationary defect. Conservation is therefore a consistency condition for the reconstructed equation, not a universal microscopic relaxation law.

A Ward operators for representative tensor fields

This appendix makes the formal-adjoint notation concrete. Boundary terms are suppressed by compact support.

A.1 Scalar

For a scalar s,

$$\mathcal{L}_\xi s = \xi^\nu \nabla_\nu s,$$

and hence

$$\mathcal{W}_\nu(P; s) = P \nabla_\nu s.$$

For a scalar density of nonzero weight the Lie derivative contains an additional divergence term; it should not be inserted into this formula as an ordinary scalar.

A.2 Covector

For v_μ, equation (19) gives

$$\mathcal{W}_\nu(P; v) = P^\mu \nabla_\nu v_\mu - \nabla_\mu (P^\mu v_\nu).$$

If $P^\mu = 0$, the contribution vanishes, but away from the vector equation it contains both P and its derivative.

A.3 Contravariant vector

For w^μ,

$$\int \sqrt{-g}\, P_\mu \mathcal{L}_\xi w^\mu = \int \sqrt{-g}\, [P_\mu \xi^\rho \nabla_\rho w^\mu - P_\mu w^\rho \nabla_\rho \xi^\mu]$$
$$= \int \sqrt{-g}\, \xi^\nu\, [P_\mu \nabla_\nu w^\mu + \nabla_\rho (P_\nu w^\rho)]\,. \tag{80}$$

Therefore

$$\mathcal{W}_\nu(P; w) = P_\mu \nabla_\nu w^\mu + \nabla_\rho (P_\nu w^\rho). \tag{81}$$

A.4 Covariant rank-two tensor

Let $q_{\mu\nu}$ have no assumed symmetry and let its Euler dual be $P^{\mu\nu}$. Since

$$\mathcal{L}_\xi q_{\mu\nu} = \xi^\rho \nabla_\rho q_{\mu\nu} + q_{\rho\nu} \nabla_\mu \xi^\rho + q_{\mu\rho} \nabla_\nu \xi^\rho,$$

integration by parts yields

$$\mathcal{W}_\lambda(P; q) = P^{\mu\nu} \nabla_\lambda q_{\mu\nu} - \nabla_\mu (P^{\mu\nu} q_{\lambda\nu}) - \nabla_\nu (P^{\mu\nu} q_{\mu\lambda}). \tag{82}$$

If P and q are symmetric, the last two terms combine after relabeling. These formulas confirm that the Ward identity is linear in the Euler expressions but can contain their derivatives.

B Boundary and distributional identities

B.1 Smooth non-null boundary

Let n^μ be the outward unit normal and $h_{\mu\nu}$ the induced metric. For a symmetric metric Euler tensor, the integration-by-parts term is

$$-2 \int_{\partial M} \sqrt{|h|}\, n^\mu X_{\mu\nu} \xi^\nu.$$

The symplectic-potential term from varying the Lagrangian must be added before assigning physical meaning to this expression. Under Dirichlet conditions for Einstein gravity, the Gibbons-Hawking-York term cancels the normal derivative of the metric variation. It does not automatically settle corner terms or other boundary conditions.

B.2 Thin interface

Suppose a hypersurface Σ divides M into M_+ and M_-. Write a distributional stress tensor as

$$T^{\mu\nu} = T_+^{\mu\nu} \Theta_+ + T_-^{\mu\nu} \Theta_- + S^{\mu\nu} \delta_\Sigma, \tag{83}$$

with $S^{\mu\nu}$ tangential when appropriate. Its distributional divergence contains:

(a) the bulk divergences in $M_\pm$;

(b) a delta contribution from the jump $n_\mu [T^\mu{}_\nu]$; and

(c) the intrinsic divergence of $S^\mu{}_\nu$, together with extrinsic curvature terms according to the projection.

The surface Ward identity balances these terms. Requiring only $\nabla_\mu T_\pm^{\mu\nu} = 0$ misses the junction condition.

B.3 Null boundary

A null hypersurface has no unit normal distinct from its tangent generator. Its induced metric is degenerate, and the non-null formula with $\sqrt{|h|}\, n^\mu$ cannot simply be continued by setting $n^2 = 0$. Null variational principles introduce a null generator, auxiliary transverse structure, expansion, and joint terms. At null infinity, radiative symplectic flux is generally nonzero. A charge balance law must include that flux, as in the Wald-Zoupas construction.

C Finite executable model

The companion Haskell modules implement a finite exact-arithmetic model of three algebraic statements used in this paper.

First, for symmetric 2×2 tensors E and T, the defect is

$$C = E - \kappa T, \tag{84}$$

where κ represents $8\pi G_{\mathrm{N}}$. A linear pullback by a matrix A is

$$A^* C = A^{\mathsf{T}} C A. \tag{85}$$

The program exhaustively checks

$$A^*(E - \kappa T) = A^* E - \kappa A^* T$$

and the composition law $(AB)^* C = B^*(A^* C)$ on a bounded integer domain.

Second, five Boolean assumptions represent the gravitational identity, matter equations, source neutrality, anomaly cancellation, and zero boundary flux. The executable checks that all five remove all listed obstructions and that removing each one restores its corresponding obstruction. This is a finite truth-table model of theorem 6.2, not a proof of the continuum identity.

Third, the code evaluates two positive diagonal mobilities on the same nonzero residual covector. Both decrease the same quadratic potential but produce different velocities. This is the exact finite-dimensional counterexample used in theorem 11.2.

The implementation uses rational arithmetic and total functions on explicit algebraic data types. It does not discretize a Lorentzian manifold, evaluate a functional determinant, establish anomaly cancellation, or prove that a microscopic reconstruction exists. Its role is limited to checking the finite identities just stated.

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