Relative Entropy, Entanglement Equilibrium, and Linearized Gravity

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Abstract

The first variation of relative entropy is a kinematic identity, whereas a gravitational field equation is a spacetime constraint. This paper gives a coefficient-complete comparison of two established bridges between them. For a ball in the vacuum of a holographic conformal field theory, the local modular Hamiltonian, the Ryu-Takayanagi or generalized-entropy dictionary, the holographic stress tensor, and an Iyer-Wald identity convert the entanglement first law for all balls into the bulk equations linearized about anti-de Sitter space. For a small geodesic ball in a local causal diamond, vacuum entanglement stationarity at fixed volume combines the CFT modular Hamiltonian with the area deficit $\delta A|_V = -\Omega_{d-2}\ell^d \delta G_{00}/(d^2-1)$. With entropy density $\eta = 1/(4\hbar G)$, the common ball moment cancels and gives the coefficient $8\pi G$. Conservation leaves a state-independent cosmological term. The two arguments are presented side by side but are not identified: the first uses an asymptotic AdS/CFT dictionary, while the second postulates a local vacuum equilibrium principle and a universal area density. We separate conformal from nonconformal matter, classical area from generalized entropy, first-order constraints from second-order canonical energy, and Einstein from higher-curvature gravity. In particular, nonconformal modular response needs an additional scalar assumption, and higher-curvature entanglement equilibrium at fixed generalized volume yields linearized constraints rather than a generic nonlinear completion. Relative-entropy positivity becomes positivity of bulk canonical energy only inside the appropriate holographic code subspace. None of these results implies that an arbitrary quantum system reconstructs a metric or obeys a nonlinear gravitational equation.

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1 Introduction

Relative entropy compares a state ρ with a reference state σ. Writing $K_\sigma = -\log\sigma$, its elementary decomposition is

$$D(\rho\|\sigma) = \operatorname{Tr}\rho(\log\rho - \log\sigma) = \Delta\langle K_\sigma\rangle - \Delta S. \tag{1}$$

Here $\Delta\langle K_\sigma\rangle = \operatorname{Tr}[(\rho-\sigma)K_\sigma]$ and $\Delta S = S(\rho) - S(\sigma)$. For a differentiable path through σ, the linear term in (1) vanishes. Thus

$$\delta S = \delta\langle K_\sigma\rangle. \tag{2}$$

This is the first law of entanglement. It follows from state-space calculus and does not mention a metric, curvature, a stress-tensor dictionary, or a gravitational action [3].

There are nevertheless two important settings in which (2) constrains gravity. The first begins with the vacuum of a conformal field theory on Minkowski space and a ball-shaped boundary region. Conformal symmetry makes the modular Hamiltonian local. In a theory with a semiclassical AdS dual, an entropy functional and the holographic stress tensor translate the two sides of the first law into bulk surface terms. Lashkari, McDermott, and Van Raamsdonk showed this directly for Einstein gravity [15]; Faulkner, Guica, Hartman, Myers, and Van Raamsdonk placed the equivalence in the Iyer-Wald framework and extended it to higher-curvature theories [7]. Requiring the first law for all boundary balls and all Lorentz frames gives the bulk gravitational equations linearized about pure AdS.

The second setting is local rather than holographic. Jacobson considered a small geodesic ball and its causal diamond in an approximately maximally symmetric spacetime [13]. The geometric input is the area deficit of the ball at fixed volume. The matter input is the modular-energy variation of a locally vacuum-like state. The physical postulate is that the total vacuum entanglement is stationary at fixed volume. For conformal matter, the relevant modular Hamiltonian is local and the same radial moment appears on the geometric and matter sides. The cancellation fixes the Einstein coefficient once the area density of entropy is identified with the Bekenstein-Hawking value.

The routes are related by conformal modular flow and covariant variational identities, but they are not the same derivation. The holographic route has a boundary CFT, a bulk asymptotic region, an AdS-Rindler wedge, and an entropy-area dictionary. The local-diamond route has no boundary dual. It instead assumes a universal short-distance area density, local maximal symmetry, and entanglement stationarity at a fixed geometric quantity. A valid comparison should expose the parallel algebra without erasing these different premises.

This distinction is especially important in four places. First, the simple ball modular Hamiltonian is exact for the vacuum of a CFT; for a generic QFT or a generic region the modular Hamiltonian is nonlocal. Second, the Ryu-Takayanagi area formula is a leading semiclassical statement. Quantum bulk entropy, time dependence, higher derivatives, and edge terms require generalized formulas. Third, both central arguments are first-order statements about perturbations of specified reference states. Positivity of relative entropy enters at second order and has a different gravitational image. Fourth, the special small-ball passage from a linearized Einstein constraint to the Einstein tensor at a point does not extend to generic higher-curvature equations.

The purpose of this paper is to make those qualifications part of the mathematics. The main results are as follows.

(i) Theorem 2.1 derives the first law and identifies its quadratic remainder with the Bogoliubov-Kubo-Mori form, preserving the information-metric convention fixed earlier in this series.

(ii) Theorem 2.3 evaluates the dimension-dependent ball moment that controls both routes.

(iii) Theorem 4.1 states the holographic all-balls implication with the Iyer-Wald normalization and the precise role of the stress and entropy dictionaries.

(iv) Theorem 5.2 derives the local-diamond coefficient, including $\hbar$, fixed volume, the conformal assumption, and the state-independent cosmological term.

(v) Theorem 6.1 states the higher-curvature extension at fixed generalized volume and its linearized boundary.

(vi) Theorem 7.1 separates the quadratic canonical-energy statement from the first-order gravitational constraint.

(vii) Theorem 8.2 gives a comparison theorem whose conclusion is equality of the resulting linearized Einstein tensor only when both independent dictionaries are simultaneously supplied.

We use metric signature $(-,+,\ldots,+)$. Unless $\hbar$ is displayed, units with $c=\hbar=1$ are used. In the holographic sections, d is the boundary spacetime dimension and the bulk has dimension $d+1$. In the local-diamond sections, d is the spacetime dimension of the diamond. Thus a spatial ball has dimension $d-1$ in both discussions. The area of the unit $(d-2)$-sphere is Ω_{d-2}.

2 Relative entropy and its perturbative orders

2.1 Reference-state conventions

Let σ be a faithful density operator and let $\rho(\lambda)$ be a twice differentiable path of faithful density operators with $\rho(0)=\sigma$ and $\operatorname{Tr}\rho(\lambda)=1$. The reference modular Hamiltonian is

$$K_\sigma = -\log\sigma. \tag{3}$$

Adding a multiple of the identity to K_σ changes neither modular flow nor any variation along normalized states. In continuum quantum field theory the reduced algebra can be of type III and a density matrix need not exist. The corresponding statements are then formulated using relative modular operators. The density-matrix notation records the perturbative algebra without claiming that the continuum factorization is literal [1].

Define

$$\Delta_\lambda \langle K_\sigma \rangle = \mathrm{Tr}[(\rho(\lambda) - \sigma) K_\sigma], \tag{4}$$

$$\Delta_\lambda S = S(\rho(\lambda)) - S(\sigma), \tag{5}$$

$$\mathcal{D}_\sigma(\lambda) = D(\rho(\lambda) \| \sigma). \tag{6}$$

Then

$$\mathcal{D}_\sigma(\lambda) = \Delta_\lambda \langle K_\sigma \rangle - \Delta_\lambda S. \tag{7}$$

The order of the arguments matters: the modular Hamiltonian in (7) belongs to the fixed second argument.

Proposition 2.1 (First order and second order)**.** *For the path above,*

$$\dot{\mathcal{D}}_\sigma(0) = 0, \tag{8}$$

$$\delta S = \delta \langle K_\sigma \rangle, \tag{9}$$

$$\ddot{\mathcal{D}}_\sigma(0) = g_\sigma^{\mathrm{BKM}}(\dot\rho_0, \dot\rho_0) \geq 0. \tag{10}$$

Consequently,

$$D(\rho(\lambda) \| \sigma) = \frac{\lambda^2}{2} g_\sigma^{\mathrm{BKM}}(\dot\rho_0, \dot\rho_0) + \mathcal{O}(\lambda^3) \tag{11}$$

when the third-order remainder exists.

Proof. The derivative of $\mathrm{Tr}(\rho \log \rho)$ at a faithful state is $\mathrm{Tr}[\dot\rho(\log \rho + 1)]$. Since $\mathrm{Tr}\,\dot\rho = 0$, differentiation of $D(\rho(\lambda) \| \sigma)$ at $\lambda = 0$ gives

$$\mathrm{Tr}[\dot\rho_0(\log \sigma - \log \sigma)] = 0. \tag{12}$$

Combining this with (7) proves the first law. The second derivative is the Frechet derivative of the logarithm,

$$g_\sigma^{\mathrm{BKM}}(X, Y) = \mathrm{Tr}\left[X (\mathrm{d}\log)_\sigma(Y)\right], \tag{13}$$

which is the BKM metric and is positive on traceless Hermitian tangents. Taylor's theorem gives (11). □

The BKM label is not optional. The Hessian of Umegaki relative entropy is not generally the Bures metric or the symmetric-logarithmic-derivative Fisher metric. This paper uses "quantum Fisher information" only when quoting literature that defines it by the relative-entropy Hessian; the actual bilinear form is identified as BKM.

Remark 2.2 (What positivity adds). The equality (9) uses only the vanishing linear term. Positivity $D \geq 0$ constrains the Hessian in (11). It cannot, without a reconstruction dictionary, turn state-space indices into spacetime indices. This order separation is the reason canonical energy appears in Section 7 rather than in the first-order derivations.

2.2 The CFT vacuum modular Hamiltonian for a ball

Let $B = B_R(\boldsymbol{x}_0)$ be the ball $|\boldsymbol{x} - \boldsymbol{x}_0| < R$ on the $t = t_0$ slice of d-dimensional Minkowski space. For the vacuum of a CFT, the domain of dependence of B is conformally related to a Rindler wedge. The modular Hamiltonian is therefore local [2, 6]:

$$K_B = 2\pi \int_B \mathrm{d}^{d-1}x \, \frac{R^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}{2R} T_{00}(t_0, \boldsymbol{x}) + c_B \mathbf{1}. \tag{14}$$

Here and through Section 4 we set $\hbar = 1$. Restoring units replaces 2π by $2\pi/\hbar$. The constant c_B normalizes the density matrix and drops out of normalized variations.

For a perturbation of the vacuum,

$$\delta\langle K_B\rangle = 2\pi \int_B \mathrm{d}^{d-1}x \, \frac{R^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}{2R} \delta\langle T_{00}(t_0, \boldsymbol{x})\rangle. \tag{15}$$

No analogous local stress-tensor expression is known for a generic region or generic reference state. Even in a CFT, locality here is tied to the vacuum ball and its conformal Killing flow.

Lemma 2.3 (The ball moment)**.** *If f is approximately constant over the ball, then*

$$\int_{|\boldsymbol{x}|<R} \mathrm{d}^{d-1}x \, \frac{R^2 - r^2}{2R} f = \frac{\Omega_{d-2} R^d}{(d-1)(d+1)} f = \frac{\Omega_{d-2} R^d}{d^2 - 1} f. \tag{16}$$

Proof. Writing $\mathrm{d}^{d-1}x = \Omega_{d-2} r^{d-2} \mathrm{d}r$ gives

$$\int_0^R \Omega_{d-2} r^{d-2} \frac{R^2 - r^2}{2R} \mathrm{d}r = \frac{\Omega_{d-2}}{2R} \left(\frac{R^{d+1}}{d-1} - \frac{R^{d+1}}{d+1} \right) \tag{17}$$

$$= \frac{\Omega_{d-2} R^d}{(d-1)(d+1)}. \tag{18}$$

□

Thus for a small ball relative to the stress-tensor variation scale,

$$\delta\langle K_B\rangle = \frac{2\pi \Omega_{d-2} R^d}{d^2 - 1} \delta\langle T_{00}(x_0)\rangle + \mathcal{O}(R^{d+1} \nabla T). \tag{19}$$

The coefficient in (19) will reappear in the local causal-diamond argument. Its appearance in both routes comes from the same conformal Killing weight, not from an identity between the two gravitational dictionaries.

3 Two dictionaries with parallel first-law algebra

It is useful to state the comparison before deriving either route. The following table records the distinct objects that occupy parallel algebraic positions.

Ingredient	Holographic boundary ball	Local causal diamond
Reference	CFT vacuum on a boundary ball; pure AdS bulk	Locally maximally symmetric vacuum of geometry and fields
Region	Boundary ball B and AdS-Rindler wedge Σ_B	Small geodesic ball Σ and its domain of dependence
Modular input	Exact local CFT vacuum ball Hamiltonian	Exact for CFT vacuum; conjectural extra scalar response for nonconformal QFT
Geometric entropy	$A(\widetilde{B})/(4G_{\mathrm{N}})$ at classical Einstein order; Wald or generalized entropy under further assumptions	Universal UV area density ηA; generalized Wald plus JKM terms for higher curvature
Stress input	Holographic stress tensor from the asymptotic metric	Local expectation value $\delta\langle T_{ab}\rangle$
Completeness	First law for all balls, centers, radii, and Lorentz frames; radial constraints	All centers and timelike frames, with a small-ball expansion
Fixed quantity	Boundary region and reference modular flow	Ordinary volume V in Einstein gravity; generalized volume W' in higher curvature
Conclusion	Bulk equation linearized about the AdS reference solution	Einstein constraint at first order about local vacuum; special pointwise Einstein reading under the small-ball assumptions
Cosmological term	Fixed by the chosen AdS background and bulk action	State-independent integration constant after Bianchi and matter Ward identities
Primary failure	No AdS/CFT entropy and stress dictionaries	No universal area density, equilibrium postulate, or local modular response

Definition 3.1 (Entanglement-gravity dictionary)**.** For a class of regions and perturbations, an entanglement-gravity dictionary consists of:

1. a specified reference state and modular generator;
2. a geometric entropy functional with fixed normalization;
3. a map from state response to a stress tensor or gravitational canonical charge;
4. a map from allowed state variations to metric and matter perturbations;
5. a completeness statement converting the family of integrated first laws into the asserted local constraint.

The definition is deliberately stronger than the entanglement first law. The latter supplies none of items 2–5. It is also route-neutral: it does not pretend that the realizations in the two columns are interchangeable.

4 The holographic ball-region route

4.1 Domain and geometric setup

Consider a d-dimensional CFT with a semiclassical $(d+1)$-dimensional AdS dual of radius L. The reference boundary state is the CFT vacuum and the reference bulk geometry is pure AdS_{d+1}. A boundary ball B determines an AdS-Rindler wedge with bifurcation surface $\widetilde{B}$ and a bulk spacelike region Σ_B satisfying

$$\partial\Sigma_B = \widetilde{B} \cup (-B). \tag{20}$$

The boundary conformal Killing vector extends to a bulk Killing vector ξ_B of pure AdS. It vanishes on $\widetilde{B}$ and has surface gravity 2π in the modular normalization.

At leading large-N order in a static Einstein bulk, the Ryu-Takayanagi dictionary is

$$S_B^{\mathrm{CFT}} = \frac{A(\widetilde{B})}{4G_{\mathrm{N}}}, \tag{21}$$

where $G_{\mathrm{N}} = G_{d+1}$. The covariant time-dependent extension is the HRT extremal-surface prescription [10]. The classical area formula was introduced by Ryu and Takayanagi [17]. In the present linearized vacuum-ball setting, the displacement of the extremal surface does not contribute at first order, because the reference surface is extremal. One may therefore evaluate the first area variation on the unperturbed hemisphere.

Put the metric in Fefferman-Graham gauge near a flat boundary:

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left[\mathrm{d}z^2 + \left(\eta_{\mu\nu} + z^d g^{(d)}_{\mu\nu}(x) + \cdots\right)\mathrm{d}x^\mu \mathrm{d}x^\nu\right]. \tag{22}$$

For Einstein gravity, a flat boundary, and a normalizable perturbation with no additional source terms at the same order, the linearized holographic stress tensor is

$$\delta\langle T^{\mathrm{CFT}}_{\mu\nu}\rangle = \frac{dL^{d-1}}{16\pi G_{\mathrm{N}}}\, g^{(d)}_{\mu\nu}. \tag{23}$$

Local counterterms and anomalies modify the expression for curved or even-dimensional boundary data, but not the flat, source-free coefficient used here. Faulkner et al. also showed that the stress coefficient can be recovered from the small-ball first law once the entropy functional is known [7].

4.2 The Iyer-Wald form

Take the Einstein bulk Lagrangian $(d+1)$-form

$$\mathbf{L} = \frac{1}{16\pi G_{\mathrm{N}}}\left(R + \frac{d(d-1)}{L^2}\right)\boldsymbol{\varepsilon}. \tag{24}$$

Its first variation is

$$\delta\mathbf{L} = E_{ab}\delta g^{ab}\boldsymbol{\varepsilon} + \mathrm{d}\boldsymbol{\Theta}(g;\delta g), \qquad E_{ab} = \frac{1}{16\pi G_{\mathrm{N}}}(G_{ab} + \Lambda g_{ab}), \tag{25}$$

with $\Lambda = -d(d-1)/(2L^2)$. Let $\boldsymbol{Q}[\xi_B]$ be the Noether-charge $(d-1)$-form. For a perturbation $h_{ab} = \delta g_{ab}$, define

$$\boldsymbol{\chi}_B(h) = \delta \boldsymbol{Q}[\xi_B] - \xi_B \cdot \boldsymbol{\Theta}(g; h). \tag{26}$$

Because ξ_B is Killing on the reference solution, the off-shell Iyer-Wald identity gives

$$\mathrm{d}\boldsymbol{\chi}_B = -2\xi_B^a \delta E_{ab}\, \boldsymbol{\varepsilon}^b. \tag{27}$$

Our definition of E_{ab} in (25) fixes the factor of two and the factor $16\pi G_\mathrm{N}$ in (27).

The two surface integrals have the dictionary

$$\int_B \boldsymbol{\chi}_B = \delta \langle K_B^{\mathrm{CFT}} \rangle, \tag{28}$$

$$\int_{\widetilde{B}} \boldsymbol{\chi}_B = \delta \left(\frac{A(\widetilde{B})}{4G_\mathrm{N}} \right) = \delta S_B^{\mathrm{CFT}}. \tag{29}$$

The first equality uses the holographic stress tensor and the boundary limit of ξ_B. The second uses $\xi_B = 0$ on the bifurcation surface and the Noether-charge expression for horizon entropy [19, 11].

Stokes' theorem now exposes the exact logical role of the first law:

$$\delta\langle K_B \rangle - \delta S_B = \int_B \boldsymbol{\chi}_B - \int_{\widetilde{B}} \boldsymbol{\chi}_B \tag{30}$$

$$= -\int_{\Sigma_B} \mathrm{d}\boldsymbol{\chi}_B = +2 \int_{\Sigma_B} \xi_B^a \delta E_{ab} \boldsymbol{\varepsilon}^b. \tag{31}$$

The CFT first law sets the left side to zero. The result is an integrated bulk constraint for every boundary ball.

Theorem 4.1 (Holographic all-balls implication)**.** *Assume:*

(a) a CFT vacuum on Minkowski space with the local ball modular Hamiltonian (14)*;*

(b) a semiclassical asymptotically AdS_{d+1} *dual whose reference solution is pure AdS;*

(c) the classical RT area dictionary and the Einstein holographic stress tensor (23)*;*

(d) normalizable perturbations in the differentiable code of states for which those dictionaries apply;

(e) the Iyer-Wald form (26)*, its boundary limits* (28)–(29)*, and the usual asymptotic conservation and tracelessness conditions;*

(f) the entanglement first law for every boundary ball, center, radius, time slice, and Lorentz frame.

Then the bulk metric perturbation satisfies

$$\delta(G_{ab} + \Lambda g_{ab}) = 0 \tag{32}$$

throughout the corresponding bulk domain. The conclusion is the Einstein equation linearized about pure AdS, not the full nonlinear equation for a finite perturbation.

Proof. Equations (31) and (9) imply

$$\int_{\Sigma_B} \xi^a_B \delta E_{ab} \boldsymbol{\varepsilon}^b = 0 \tag{33}$$

for all balls. On a fixed boundary time slice, differentiating the ball transform with respect to its radius converts (33) into unweighted integrals over all associated bulk regions. The corresponding Radon-type injectivity implies the time-time component vanishes. Boosting the boundary ball through all Lorentz frames gives every boundary-tangent component. The linearized Noether identity $\nabla^a \delta E_{ab} = 0$, together with the asymptotic stress-tensor constraints, propagates the radial constraints and gives the remaining components. Hence $\delta E_{ab} = 0$. Substitution of (25) gives (32). This is the converse first-law argument of [7]. □

The all-balls premise is essential. A first law for one ball gives one weighted integral constraint. Even all balls on one time slice initially probe only timelike contractions. Lorentz-frame completeness, the Noether identity, and boundary constraint data are what promote those integrals to every local component.

4.3 Matter and generalized entropy

At the next order in the bulk semiclassical expansion, boundary entropy is not exhausted by classical area. For a suitable bulk effective theory and state, the FLM formula has the schematic form

$$S_B^{\mathrm{CFT}} = \frac{\langle \widehat{A}(\widetilde{B}) \rangle}{4G_{\mathrm{N}}} + S_{\mathrm{bulk}}(\Sigma_B) + S_{\mathrm{ct}}(\widetilde{B}) + \mathcal{O}(G_{\mathrm{N}}), \tag{34}$$

where local counterterm or Wald-like contributions depend on the bulk effective action [8]. The generalized entropy is the sum of the renormalized geometric and bulk entropy pieces.

Jafferis, Lewkowycz, Maldacena, and Suh gave the corresponding modular operator relation in a semiclassical code subspace,

$$K_B^{\mathrm{bdy}} = \frac{\widehat{A}(\widetilde{B})}{4G_{\mathrm{N}}} + K_{\Sigma_B}^{\mathrm{bulk}} + \text{local surface operators} + \mathcal{O}(G_{\mathrm{N}}), \tag{35}$$

and the leading equality of boundary and bulk relative entropies

$$D_B^{\mathrm{bdy}}(\rho \| \sigma) = D_{\Sigma_B}^{\mathrm{bulk}}(\rho \| \sigma) + \mathcal{O}(G_{\mathrm{N}}) \tag{36}$$

for nearby semiclassical states [14]. Local surface operators cancel between modular energy and entropy in relative entropy.

With these stronger assumptions, the first-law identity includes the bulk matter modular Hamiltonian. The corresponding linearized semiclassical constraint is

$$\delta(G_{ab} + \Lambda g_{ab}) = 8\pi G_{\mathrm{N}}\, \delta \langle T_{ab}^{\mathrm{bulk}} \rangle, \tag{37}$$

with renormalized couplings and the counterterms required by (34). Equation (37) is not obtained by inserting an arbitrary matter entropy into RT. It requires the bulk effective theory, generalized entropy, the code-subspace modular dictionary, and a consistent gravitational constraint analysis.

Remark 4.2 (RT, HRT, and generalized entropy). RT applies directly to classical static configurations. HRT supplies the covariant classical extremal surface. FLM and related generalized-entropy formulas include bulk quantum entropy. In higher-derivative gravity, the entropy of a generic nonstationary surface is not generally the naive Wald functional; extrinsic-curvature terms and ambiguity resolutions matter. For the vacuum ball, the reference surface is a bifurcation surface, and the first variation can be represented by the appropriate Wald or generalized functional under the assumptions stated in the cited derivations.

4.4 Failure domain of the holographic implication

Theorem 4.1 does not apply in any of the following cases without new input:

(1) a nonholographic QFT for which no bulk entropy or stress dictionary is supplied;

(2) a generic region whose vacuum modular Hamiltonian is nonlocal;

(3) a state outside the semiclassical code subspace;

(4) finite perturbations for which the linearized Iyer-Wald identity is insufficient;

(5) bulk boundary conditions that invalidate the normalizable Fefferman-Graham expansion or its conserved stress tensor;

(6) quantum orders at which the generalized entropy, surface operators, or gravitational dressing have not been controlled;

(7) a higher-curvature action paired with the Einstein area functional instead of its correct entropy functional;

(8) too small a family of balls to invert the integrated constraints.

Relative entropy remains well-defined in many of these cases. What fails is the gravitational dictionary or its completeness, not the state-space identity.

5 The local causal-diamond route

5.1 Small-ball geometry at fixed volume

Let p be a point in a d-dimensional spacetime and let u^a be a unit timelike vector at p. Shoot spatial geodesics orthogonal to u^a to form a geodesic ball Σ of radius ℓ. Assume

$$\ell \ll L_{\mathrm{curv}}, \tag{38}$$

and work to first order in curvature relative to a locally maximally symmetric reference geometry.

For orientation, first compare to flat space. Let $\mathcal{R} = R_{ij}{}^{ij}$ be the spatial Ricci scalar in Riemann normal coordinates adapted to u^a. The volume and area variations at fixed geodesic radius are

$$\delta V|_\ell = -\frac{\Omega_{d-2}\ell^{d+1}}{6(d-1)(d+1)}\,\mathcal{R}, \tag{39}$$

$$\delta A|_\ell = -\frac{\Omega_{d-2}\ell^d}{6(d-1)}\,\mathcal{R}. \tag{40}$$

Holding volume rather than radius fixed requires the compensating radial variation

$$\delta A|_V = \delta A|_\ell - \frac{d-2}{\ell}\delta V|_\ell. \tag{41}$$

Since $\mathcal{R} = 2G_{ab}u^a u^b$ at the center, the result is

$$\boxed{\delta A|_V = -\frac{\Omega_{d-2}\ell^d}{d^2-1}\delta G_{ab}u^a u^b.} \tag{42}$$

For variation away from a maximally symmetric reference satisfying $G^{(0)}_{ab} = -\lambda g^{(0)}_{ab}$, the left side becomes

$$\delta A|_{V,\lambda} = -\frac{\Omega_{d-2}\ell^d}{d^2-1}(G_{ab} + \lambda g_{ab})u^a u^b, \tag{43}$$

where the parenthesis is already first order relative to the reference because it vanishes in that reference.

The fixed-volume condition is not cosmetic. Using (40) instead would retain the spatial Ricci scalar rather than the Einstein contraction and would change the coefficient. In Jacobson's calculation, it would miss the required result by a factor depending on d [13].

5.2 Conformal modular response with units restored

The flat-diamond conformal Killing vector can be normalized to unit surface gravity. On the central slice $t = 0$ it is

$$\zeta^a = \frac{\ell^2 - r^2}{2\ell}u^a. \tag{44}$$

For a CFT vacuum, the dimensionless modular Hamiltonian is

$$K_\Sigma = \frac{2\pi}{\hbar}\int_\Sigma T_{ab}u^a\zeta^b \mathrm{d}V + c_\Sigma \mathbf{1}. \tag{45}$$

If the excitation wavelength is large compared with the ball,

$$\ell \ll L_{\mathrm{exc}}, \tag{46}$$

then $\delta\langle T_{ab}\rangle$ is approximately constant and Theorem 2.3 yields

$$\delta S_{\mathrm{IR}} = \delta\langle K_\Sigma\rangle = \frac{2\pi}{\hbar}\frac{\Omega_{d-2}\ell^d}{d^2-1}\delta\langle T_{ab}\rangle u^a u^b + \mathcal{O}(\ell^{d+1}\nabla T). \tag{47}$$

The short-distance geometric contribution is postulated to have a universal, state-independent entropy density η in a chosen conformal frame:

$$\delta S_{\mathrm{UV}} = \eta\,\delta A. \tag{48}$$

The total first variation at fixed volume is therefore

$$\delta S_{\mathrm{tot}}|_{V,\lambda} = \eta\,\delta A|_{V,\lambda} + \delta S_{\mathrm{IR}}. \tag{49}$$

Assumption 5.1 (Local entanglement equilibrium)**.** For every point, every unit timelike vector, and sufficiently small geodesic balls, the locally maximally symmetric vacuum is stationary under simultaneous first-order variations of geometry and the CFT state at fixed ball volume:

$$\delta S_{\rm tot}|_{V,\lambda} = 0. \tag{50}$$

The area density η is finite, positive, and universal over the class of states being compared.

This is a physical postulate, not a consequence of relative entropy alone. The matter first law determines $\delta S_{\rm IR}$ once the modular Hamiltonian is known. It does not assert that the geometric entropy is ηA, that volume must be fixed, or that the combined entropy is stationary.

Theorem 5.2 (Local-diamond coefficient and Einstein constraint)**.** *Assume Theorem 5.1, the small-ball conditions* (38) *and* (46)*, conformal matter with modular response* (47)*, and the fixed-volume geometric identity* (43)*. Then for every unit timelike u^a,*

$$(G_{ab} + \lambda g_{ab}) u^a u^b = \frac{2\pi}{\hbar\eta} \delta\langle T_{ab}\rangle u^a u^b. \tag{51}$$

Consequently the tensor relation is

$$G_{ab} + \lambda g_{ab} = \frac{2\pi}{\hbar\eta} \delta\langle T_{ab}\rangle. \tag{52}$$

If

$$\eta = \frac{1}{4\hbar G}, \tag{53}$$

then

$$G_{ab} + \lambda g_{ab} = 8\pi G\, \delta\langle T_{ab}\rangle. \tag{54}$$

Using the geometric Bianchi identity and the anomaly-free matter Ward identity, integration over state variations leaves

$$G_{ab} + \Lambda g_{ab} = 8\pi G\langle T_{ab}\rangle, \tag{55}$$

where Λ is independent of the state and constant on each connected spacetime component.

Proof. Substitute (43) and (47) into (50). Both terms contain the nonzero common factor

$$\frac{\Omega_{d-2}\ell^d}{d^2-1}. \tag{56}$$

After cancellation,

$$-\eta\, (G_{ab} + \lambda g_{ab}) u^a u^b + \frac{2\pi}{\hbar} \delta\langle T_{ab}\rangle u^a u^b = 0. \tag{57}$$

This is (51). A symmetric tensor whose contraction with every timelike unit vector vanishes is zero, so polarization gives (52). Equation (53) gives

$$\frac{2\pi}{\hbar\eta} = 8\pi G, \tag{58}$$

which proves (54).

The relation is first order relative to the local vacuum even though the Einstein tensor at the ball center is written without a delta. The contracted Bianchi identity gives $\nabla^a G_{ab} = 0$. The diffeomorphism Ward identity, assuming matter equations and no uncanceled anomaly, gives $\nabla^a \langle T_{ab} \rangle = 0$. Metric compatibility therefore implies $\nabla_b \lambda = 0$ in the conformal case. Writing the resulting state-independent constant as Λ proves (55). Neither identity determines the numerical value of Λ. □

Remark 5.3 (State independence of the cosmological term). The cosmological term is not suppressed by long distance and is not produced by the ball moment. The local-diamond derivation fixes changes between nearby states. A term common to all those states is invisible to the variation. Bianchi and Ward identities make that term spacetime constant under the stated assumptions, but a separate boundary condition or vacuum selection is needed to fix its value. In the holographic route, by contrast, the reference AdS radius and bulk action already specify the background cosmological term.

5.3 Conformal versus nonconformal matter

For a generic QFT and a finite ball, K_Σ is not the local stress integral (45). Jacobson proposed a short-distance form for a QFT approaching a UV fixed point. If

$$\ell \ll L_{\mathrm{QFT}}, \tag{59}$$

one may conjecture, to leading order in ℓ, that

$$\delta\langle K_\Sigma \rangle = \frac{\Omega_{d-2}\ell^d}{d^2-1}\left(\delta\langle T_{ab}\rangle u^a u^b + \delta X\right), \tag{60}$$

where the normalization of K here follows Jacobson's convention in which $\delta S = (2\pi/\hbar)\delta\langle K \rangle$. Equivalently, in the dimensionless convention of (45), multiply the entire right side by $2\pi/\hbar$. The scalar δX can depend on the state, the QFT deformation, and the ball scale.

The equilibrium equation becomes

$$G_{ab} + \lambda g_{ab} = \frac{2\pi}{\hbar\eta}\left(\delta\langle T_{ab}\rangle + \delta X g_{ab}\right). \tag{61}$$

Conservation relates the local reference curvature to δX:

$$\lambda = \frac{2\pi}{\hbar\eta}\delta X + \Lambda. \tag{62}$$

Substitution recovers (55). This cancellation is conditional on (60). If δX has uncontrolled spacetime or ℓ dependence, dominates the stress contribution, or is not the only nonlocal modular correction, the Einstein inference does not follow. Studies of relevant deformations show that such terms require case-by-case control [5, 18].

Thus the conformal result is a derivation from a known local modular Hamiltonian, whereas the nonconformal extension imports an additional short-distance conjecture. They should not be combined under an unqualified phrase such as "the small-ball first law."

5.4 How much nonlinearity is obtained?

The entropy calculation is first order about a local vacuum. Jacobson's Einstein-gravity argument has a special additional observation: in Riemann normal coordinates the leading metric departure from flat space contains the actual curvature tensor at the center, and the Einstein tensor is linear in curvature. Repeating the first-order equilibrium condition in arbitrarily small balls centered at every point can therefore be read as the Einstein equation evaluated at each center, within the scale window

$$\ell_{\rm UV} \ll \ell \ll \min(L_{\rm curv}, L_{\rm exc}, L_{\rm QFT}). \tag{63}$$

This is not a generic derivation of nonlinear dynamics from $\delta S = \delta\langle K\rangle$. It depends on the linearity of the Einstein tensor in curvature, the local maximally symmetric comparison, and the ability to impose the hypothesis at every point and frame. Finite state changes, backreaction beyond the local expansion, higher-order entropy variations, and global boundary conditions are not controlled by the first-order stationarity statement. The limitation becomes decisive for higher-curvature actions in Section 6.

5.5 Relation to the 1995 local-horizon equation of state

Jacobson's earlier argument uses local Rindler horizons rather than compact causal diamonds [12]. Choose a small patch of a local causal horizon with approximate boost Killing vector $\chi^a = -\kappa\lambda k^a$ along affinely parameterized null generators k^a. The heat flux is

$$\delta Q = \int T_{ab}\chi^a \mathrm{d}\Sigma^b = -\kappa \int \lambda T_{ab} k^a k^b \mathrm{d}\lambda \mathrm{d}A. \tag{64}$$

The Unruh temperature is

$$T = \frac{\hbar\kappa}{2\pi}. \tag{65}$$

Assuming horizon entropy variation $\delta S = \eta\delta A$, the Raychaudhuri equation with initially vanishing expansion and shear gives

$$\delta A = -\int \lambda R_{ab} k^a k^b \mathrm{d}\lambda \mathrm{d}A \tag{66}$$

to leading order. Imposing $\delta Q = T\delta S$ for all null vectors yields

$$R_{ab} + \Phi g_{ab} = \frac{2\pi}{\hbar\eta} T_{ab}. \tag{67}$$

Bianchi and conservation fix $\Phi = -R/2 + \Lambda$, producing Einstein's equation with the same coupling when (53) holds.

The 1995 construction is a horizon Clausius argument involving null energy flux and Raychaudhuri focusing. The 2016 construction is a compact-region entanglement-equilibrium argument involving a reduced state, modular energy, and fixed spatial volume. The shared coefficient and thermodynamic language do not make their assumptions identical. In particular, no heat flux crosses the central slice of the equilibrium ball.

6 Higher-curvature entanglement equilibrium

6.1 Covariant Lagrangians and Wald entropy

Let the gravitational Lagrangian be a diffeomorphism-invariant d-form $\mathbf{L} = \mathcal{L}\boldsymbol{\varepsilon}$, depending locally on the metric, curvature, and possibly symmetrized derivatives of curvature. For the simpler $\mathcal{L}(g, R_{abcd})$ case define

$$E^{abcd} = \frac{\partial \mathcal{L}}{\partial R_{abcd}}. \tag{68}$$

On a maximally symmetric background,

$$E^{abcd}_{(0)} = E_0(g^{ac}g^{bd} - g^{ad}g^{bc}) \tag{69}$$

for a constant E_0. The Wald entropy of the diamond edge is

$$S_{\mathrm{Wald}} = -2\pi \int_{\partial\Sigma} \mu\, E^{abcd} n_{ab} n_{cd}, \tag{70}$$

where n_{ab} is the binormal and μ is the induced area element.

For a causal diamond in a maximally symmetric background, Bueno, Min, Speranza, and Visser define the generalized volume

$$W = \frac{1}{(d-2)E_0} \int_{\Sigma} \eta \left(E^{abcd} u_a u_d h_{bc} - E_0 \right), \tag{71}$$

where $h_{ab} = g_{ab} + u_a u_b$ is the spatial metric and η its volume form [4]. With the normalization in (71), W reduces to the ordinary volume in Einstein gravity, for which $E_0 = 1/(32\pi G)$.

The covariant phase space has Jacobson-Kang-Myers ambiguities. A change of symplectic potential by an exact form shifts the entropy and generalized volume by surface terms. Write the corresponding, entropy-compatible combination as

$$S_{\mathrm{grav}} = S_{\mathrm{Wald}} + S_{\mathrm{JKM}}, \qquad W' = W + W_{\mathrm{JKM}}. \tag{72}$$

The same ambiguity resolution must be used in both terms.

6.2 Off-shell causal-diamond identity

Let ζ^a be the conformal Killing vector of the reference diamond, with surface gravity κ, and let H^{m}_{ζ} be the matter Hamiltonian generating its flow. The higher-curvature off-shell identity is

$$\frac{\kappa}{2\pi} \delta S_{\mathrm{grav}} \Big|_{W'} + \delta H^{\mathrm{m}}_{\zeta} = \int_{\Sigma} \delta \boldsymbol{C}_{\zeta}, \tag{73}$$

where $\delta \boldsymbol{C}_{\zeta} = 0$ is the linearized gravitational constraint. For conformal matter, the first law identifies $\delta H^{\mathrm{m}}_{\zeta}$ with the matter entanglement variation in the same normalization. The total entanglement variation at fixed W' is then

$$\frac{\kappa}{2\pi} \delta S_{\mathrm{EE}} \Big|_{W'} = \int_{\Sigma} \delta \boldsymbol{C}_{\zeta}. \tag{74}$$

Theorem 6.1 (Higher-curvature equilibrium). *Suppose a diffeomorphism-invariant gravitational effective theory is expanded about a maximally symmetric solution, its geometric entanglement terms are represented by* (72)*, the causal-diamond identity* (73) *holds, and the matter sector is conformal with the local diamond modular Hamiltonian. Then*

$$\delta S_{\mathrm{EE}}\Big|_{W'} = 0 \quad \textit{for every small ball and timelike frame} \tag{75}$$

is equivalent to the linearized gravitational constraints

$$\delta \boldsymbol{C}_\zeta = 0. \tag{76}$$

It does not, for a generic higher-curvature action, imply the full nonlinear field equations at the center of the ball.

Proof. Equation (74) proves that the linearized constraint implies stationary entropy. Conversely, stationarity for all balls makes every weighted timelike constraint integral vanish. Varying centers, radii, and timelike frames gives the local linearized constraints by the same completeness logic as in the Einstein case. The conclusion remains linearized because (73) is a first variation about the maximally symmetric reference.

For a Lagrangian nonlinear in curvature, the exact field equation contains terms schematically of the form R^n and $\nabla\nabla R^{n-1}$. Linearizing about a reference curvature $\bar{R}$ retains only $n\bar{R}^{n-1}\delta R$ and corresponding derivative terms. In flat space, many algebraic higher-curvature contributions vanish at linear order. Higher terms in the Riemann normal-coordinate expansion that could recover them enter at the same order as nonlinear corrections to the linearized constraint. The first-order small-ball calculation therefore cannot separate or reconstruct the full nonlinear tensor. This is precisely the limitation established in [4]. □

Remark 6.2 (Holographic and local higher curvature). Faulkner et al. show that, for holographic CFT balls, a suitable Wald functional and holographic stress tensor lead to the equations linearized about AdS for the associated higher-curvature action [7]. Bueno et al. show that local-diamond equilibrium requires fixed generalized volume and a matched JKM ambiguity [4]. Both are linearized statements, but the regions, reference data, and completeness arguments remain different.

6.3 Effective-field-theory reading

If higher-curvature terms are controlled corrections to Einstein gravity, their linearized tensors are meaningful within the derivative expansion. For example,

$$\Gamma_{\mathrm{g}} = \int \mathrm{d}^d x \sqrt{-g}\left[\frac{R-2\Lambda}{16\pi G} + \alpha_1 R^2 + \alpha_2 R_{ab}R^{ab} + \alpha_3 R_{abcd}R^{abcd} + \cdots\right]. \tag{77}$$

At curvature length L_{curv}, the local correction is relatively small only if combinations such as

$$16\pi G|\alpha_i| L_{\mathrm{curv}}^{-2} \ll 1 \tag{78}$$

in the chosen dimension and normalization. Large Wilson coefficients, additional light modes, or massless nonlocal form factors can invalidate the truncation. Entanglement equilibrium does not remove those effective-field- theory qualifications established in the preceding parts of this series.

7 Relative-entropy positivity and canonical energy

The first law exhausts the linear term in $D(\rho_B(\lambda)\|\rho_B(0))$. The next term is the BKM Hessian:

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} D(\rho_B(\lambda)\|\rho_B(0))\bigg|_{\lambda=0} = g_B^{\mathrm{BKM}}(\delta\rho_B, \delta\rho_B). \tag{79}$$

In a holographic code subspace and for a ball whose entanglement wedge has a Killing flow, this quantity has a bulk canonical-energy representation [16].

Let $\boldsymbol{\omega}(g; h_1, h_2)$ be the covariant symplectic current of the bulk gravitational theory and let ξ_B be the AdS-Rindler Killing vector. For linearized solutions in a suitable Hollands-Wald gauge [9], define

$$\mathcal{E}_B(h_1, h_2) = \int_{\Sigma_B} \boldsymbol{\omega}(g; h_1, \mathcal{L}_{\xi_B} h_2). \tag{80}$$

Gauge transformations that preserve the asymptotic and extremal-surface data are null directions after the constraints and boundary terms are handled.

Proposition 7.1 (Canonical energy is the holographic BKM Hessian)**.** *For perturbations within the semiclassical holographic code subspace, obeying the linearized bulk equations and the boundary and gauge conditions needed by the covariant phase-space construction,*

$$g_B^{\mathrm{BKM}}(\delta\rho_B, \delta\rho_B) = \mathcal{E}_B(h, h) \tag{81}$$

at leading classical order. Hence relative-entropy positivity implies

$$\mathcal{E}_B(h, h) \geq 0. \tag{82}$$

Proof. Differentiate the boundary relative entropy twice. Its first derivative is the Iyer-Wald boundary difference in (31). Differentiating the covariant phase-space identity along a family of on-shell bulk solutions produces the symplectic flux (80); the first-order constraint terms vanish and the extremal-surface gauge fixes the residual boundary terms. This yields (81). Positivity of Umegaki relative entropy then gives (82). The detailed gauge construction is given in [16]. □

At quantum order, JLMS equality (36) identifies the boundary relative entropy with bulk relative entropy rather than with a purely classical metric quadratic form. Matter and graviton contributions must then be retained together. The positivity statement is still powerful, but its bulk interpretation changes with perturbative order.

Three nonclaims are immediate.

(1) Canonical-energy positivity does not by itself reconstruct a bulk metric. The code-subspace and region dictionary is already assumed.

(2) Positivity does not select a unique nonlinear bulk action. Different consistent theories can have positive canonical energy in their stable sectors.

(3) The local-diamond first-order argument does not automatically inherit (81). A separate equality between its state-space Hessian and a gravitational symplectic form would be required.

8 A conditional comparison theorem

The preceding derivations can now be compared without identifying their premises.

Definition 8.1 (Common Einstein perturbation)**.** A perturbation is common to the two routes if there is a specified reconstruction map that identifies:

1. a boundary CFT vacuum-ball perturbation and its bulk metric perturbation h_{ab};
2. a local causal-diamond state and geometry perturbation in the same bulk neighborhood;
3. the stress-tensor expectation values under the holographic and local descriptions;
4. the Newton coupling and background cosmological curvature used by both descriptions.

The definition is a compatibility hypothesis. Neither the boundary CFT first law nor local entanglement equilibrium constructs this identification on its own.

Theorem 8.2 (Agreement without conflation)**.** *Let a common perturbation in the sense of Theorem 8.1 satisfy all hypotheses of the generalized holographic relation* (37) *and all conformal-matter hypotheses of Theorem 5.2. Suppose the local area density obeys $\eta = 1/(4\hbar G)$ and the holographic Newton coupling is the same G. Then both routes impose the same local tensor constraint*

$$\delta(G_{ab} + \Lambda g_{ab}) = 8\pi G\, \delta\langle T_{ab}\rangle \tag{83}$$

on their shared domain. This agreement does not imply that either route's dictionary follows from the other.

Proof. The holographic dictionary and all-balls completeness give (37). The local-diamond fixed-volume calculation and entropy density give (54). The common-perturbation hypothesis identifies the tensors, couplings, background, and matter expectation values, so the two conclusions coincide as (83). Every antecedent before this final identification remains route-specific. Logical agreement of the conclusions therefore does not derive RT from local entropy density, local equilibrium from AdS/CFT, or either reconstruction map from relative entropy. □

Corollary 8.3 (Failure of the dictionary-free implication)**.** *The entanglement first law $\delta S = \delta\langle K\rangle$ and positivity $D \geq 0$, without an entanglement-gravity dictionary in the sense of Theorem 3.1, do not imply* (83)*.*

Proof. Take any faithful finite-dimensional quantum system with a differentiable state path. Proposition 2.1 and relative-entropy positivity hold. Choose no map from its state parameters to a Lorentzian metric, no geometric entropy functional, and no stress dictionary. The premises of a spacetime tensor equation are then absent while the information identities remain true. □

9 Relation to infrared reconstruction and effective action

The earlier parts of this series separate three levels of structure. A faithful quantum state family carries the BKM metric when Umegaki relative entropy is used. A reconstruction interface must additionally supply localization, probes, identifiability, and causal data. An infrared metric effective theory then requires locality, diffeomorphism redundancy, variational completeness, anomaly control, scale separation, and a specified long-range spectrum.

The present paper adds neither a generic reconstruction map nor a replacement for those infrared assumptions. Instead it studies a narrower class in which an entropy dictionary is known or postulated. The relation between the effective-action and entanglement routes is

$$\delta D = 0 \text{ at first order} \quad + \quad \text{entanglement-gravity dictionary} \quad \Longrightarrow \quad \text{linearized constraint.} \tag{84}$$

The middle term contains the real gravitational input. In the holographic case it includes RT or generalized entropy, the stress tensor, and the all-balls Iyer-Wald completeness argument. In the diamond case it includes universal area density, fixed-volume stationarity, the local modular form, and all-frame small-ball completeness.

By contrast, effective-action stationarity has the form

$$(D\mathfrak{R})^* \left[G_{ab} + \Lambda g_{ab} + H_{ab} - 8\pi G T_{ab}\right] = 0. \tag{85}$$

Only reconstruction-tangent completeness promotes (85) to the full local field equation. The all-balls completeness in Theorem 4.1 and the all-frame condition in Theorem 5.2 play an analogous functional role, but they test different spaces of variations. No equality between those completeness statements should be assumed without a common reconstruction map.

The effective-action route naturally accommodates a full nonlinear equation once its action and variational domain are supplied. The entanglement routes studied here naturally produce linearized constraints about reference states. Agreement at first order is a consistency check on the dictionaries. It is not a proof that state-space relative entropy replaces the nonlinear effective action.

10 Failure domains and adversarial cases

The most useful boundary of a conditional theorem is a list of premises that can fail independently.

10.1 No Lorentzian reconstruction

A spin chain, tensor product, or finite matrix algebra can satisfy the relative-entropy first law and positivity while having no smooth manifold, causal structure, stress tensor, or massless spin-two excitation. Neither route applies. A graph inferred from mutual information does not by itself supply the missing Lorentzian or dynamical data.

10.2 Wrong reference state or region

The local stress representation (14) is tied to a CFT vacuum ball. For an excited reference state, a deformed region, or a generic QFT, the modular Hamiltonian may contain nonlocal operators. Replacing it by the vacuum stress integral is then an uncontrolled approximation. The first law remains true with the actual K_σ, but the gravitational conversion can fail.

10.3 Rank or domain singularities

If a density-matrix path leaves the support of the reference state, $D(\rho\|\sigma)$ may diverge and K_σ may be unbounded on the new support. In continuum algebras, modular domains require separate control. The differentiability premise of Theorem 2.1 must be checked rather than inferred from formal traces.

10.4 No entropy-area dictionary

An area law in a microscopic system is not automatically the Ryu-Takayanagi formula and does not fix the coefficient $1/(4G)$. RT is a holographic dictionary. Jacobson's ηA is a universality postulate about short-distance vacuum entanglement in a gravitational theory. Systems with area-law entanglement but no propagating metric provide direct counterexamples to a dictionary-free inference.

10.5 No stress dictionary or anomalous Ward identity

Boundary stress response must match the Fefferman-Graham coefficient in the holographic route. Local matter response must be conserved in the diamond route. With an uncanceled diffeomorphism anomaly,

$$\nabla^a \langle T_{ab} \rangle = \mathcal{A}_b, \tag{86}$$

the step fixing a constant cosmological term fails unless inflow or additional degrees of freedom cancel $\mathcal{A}_b$.

10.6 Additional long-range fields

Scalar-tensor, vector-tensor, torsional, or multiple-metric theories can have additional contributions at the same derivative order. Their modular response may involve operators beyond T_{ab}, and their gravitational entropy may depend on additional fields. Forcing the result into a pure Einstein equation discards physical infrared data.

10.7 Insufficient region completeness

One boundary ball or one local timelike frame gives an integrated scalar condition, not a tensor equation. Restricted families can leave nonzero constraint tensors in the kernel of every available integral. The radius, center, frame, boundary conservation, and radial propagation conditions in the holographic proof are substantive hypotheses.

10.8 No scale window

The local-diamond calculation needs (63). If the ball approaches the UV cutoff, the assumed area density and semiclassical geometry are uncontrolled. If it approaches the curvature, excitation, or deformation scale, the constant-stress and Riemann-normal-coordinate expansions fail. A theory with no interval between these scales has no domain for the derivation.

10.9 Nonconformal modular dominance

Relevant deformations can produce scalar or nonlocal contributions that scale differently with ℓ from the CFT stress term. If the proposed δX in (60) is not controlled, cannot be absorbed into a local reference curvature, or varies incompatibly with conservation, local entropy stationarity does not yield Einstein's equation.

10.10 Uncontrolled higher curvature and nonlocality

Large higher-curvature coefficients invalidate the Einstein truncation. Massless loops generate nonanalytic, nonlocal form factors that are not captured by a finite Wald functional. Fixed generalized volume controls the linearized local higher-curvature constraint under its assumptions; it does not prove that every nonlocal effective action has a corresponding local entropy equilibrium principle.

10.11 Linearized data used as nonlinear data

The equations derived from the first law are variations about specified references. Substituting a finite metric difference for h_{ab}, retaining quadratic curvature terms while dropping second-order entropy terms, or iterating the linearized equation without controlling the reference family mixes perturbative orders. Higher relative-entropy derivatives and backreaction would be needed for a systematic nonlinear analysis.

11 Discussion

The two derivations share a striking numerical core. The conformal Killing weight $(R^2 - r^2)/(2R)$ integrates to $\Omega_{d-2}R^d/(d^2-1)$. In the holographic argument, that weight defines the boundary modular charge and the AdS-Rindler Killing vector entering the Iyer-Wald form. In the local argument, it defines the diamond modular energy, while fixed-volume geometry produces exactly the same dimension-dependent factor multiplying the Einstein tensor. The coefficient $8\pi G$ follows because the modular temperature contributes $2\pi/\hbar$ and the area entropy density contributes $1/(4\hbar G)$.

The shared factor is not a universal formula for arbitrary entangling surfaces. It is a consequence of ball geometry and conformal modular flow. For a generic region, the modular generator changes; for a generic QFT, it becomes nonlocal; for a generic gravitational action, the geometric entropy and the fixed quantity change. The coefficient audit therefore clarifies both the power and the narrowness of the result.

The cosmological term illustrates another difference. A perturbative AdS calculation begins with a bulk Lagrangian and a reference radius, so the background Λ is already present. The local equilibrium argument compares nearby states and cannot detect an additive state-independent metric term. Conservation promotes that ambiguity to a spacetime constant. This is an integration constant, not a prediction that the cosmological constant vanishes or is small.

The higher-curvature extension shows why "entanglement implies gravity" is too coarse a slogan. What the local identity controls is a linearized constraint associated with a particular Lagrangian entropy and generalized volume. The entropy functional contains theory-dependent information. The argument tests consistency between that information and the linearized field equation; it does not select the Lagrangian from the bare first law.

Second-order relative entropy gives a complementary diagnostic. In holography it equals canonical energy and therefore tests perturbative stability. The equality is a concrete realization of a relation between microscopic distinguishability and gravitational stiffness, but it already assumes the code-subspace reconstruction. The general problem of deriving that reconstruction and its Newton coupling from microscopic data remains open.

There are several directions in which the comparison can be sharpened. One is an operator-algebraic treatment that avoids factorized density matrices and controls modular domains. Another is a systematic nonconformal expansion identifying when the scalar δX is local and subleading. A third is a quantum covariant-phase-space account of generalized entropy beyond leading semiclassical order. A fourth is a model-specific test of whether the holographic and local-diamond dictionaries can be realized on the same reconstructed bulk state family, as required by Theorem 8.1.

12 Conclusion

Relative entropy supplies an exact decomposition, $D = \Delta\langle K\rangle - \Delta S$. Its first variation gives the entanglement first law, and its second variation gives the BKM metric. These are state-space facts. Gravitational consequences arise only after additional dictionaries identify modular response, geometric entropy, stress response, and a complete family of spacetime constraints.

For holographic CFT vacuum balls, the RT or generalized-entropy formula, the holographic stress tensor, and the Iyer-Wald form convert the first law for all balls into the bulk equation linearized about AdS. At quantum order, the FLM and JLMS structures are required. Relative-entropy positivity then controls bulk canonical energy inside the semiclassical code subspace.

For local causal diamonds, fixed-volume geometry gives $\delta A|_V = -\Omega_{d-2}\ell^d\delta G_{00}/(d^2-1)$. The conformal modular Hamiltonian gives the same ball moment multiplying $(2\pi/\hbar)\delta\langle T_{00}\rangle$. Entanglement stationarity and $\eta = 1/(4\hbar G)$ yield the coefficient $8\pi G$. Bianchi and Ward identities leave a state-independent cosmological term. Nonconformal matter requires an additional modular-response conjecture.

In higher-curvature gravity, entropy equilibrium must be formulated at fixed generalized volume with matched Wald and JKM terms. The result is equivalent to linearized higher-curvature constraints, not to a generic nonlinear completion. The holographic and local routes can agree on a common linearized Einstein perturbation, but only after a common reconstruction map identifies their otherwise distinct data. The agreement is therefore a conditional consistency result, not a universal derivation of spacetime from quantum information.

A Fixed-volume coefficient in detail

This appendix derives (42) with all dimension factors. On the spatial slice through the center of the ball, Riemann normal coordinates give

$$h_{ij}(x) = \delta_{ij} - \frac{1}{3}\mathcal{R}_{ikjl}x^k x^l + \mathcal{O}(r^3/L^3). \tag{87}$$

The determinant is

$$\sqrt{h} = 1 - \frac{1}{6}\mathcal{R}_{kl}x^k x^l + \mathcal{O}(r^3/L^3). \tag{88}$$

Rotational averaging uses

$$\int \mathrm{d}\Omega_{d-2}\, n^k n^l = \frac{\Omega_{d-2}}{d-1}\delta^{kl}. \tag{89}$$

Therefore

$$\delta V|_\ell = -\frac{1}{6}\frac{\Omega_{d-2}}{d-1}\mathcal{R}\int_0^\ell r^d \mathrm{d}r \tag{90}$$

$$= -\frac{\Omega_{d-2}\ell^{d+1}}{6(d-1)(d+1)}\mathcal{R}. \tag{91}$$

Differentiation with respect to ℓ gives

$$\delta A|_\ell = -\frac{\Omega_{d-2}\ell^d}{6(d-1)}\mathcal{R}. \tag{92}$$

The flat volume and area are

$$V_0 = \frac{\Omega_{d-2}}{d-1}\ell^{d-1}, \qquad A_0 = \Omega_{d-2}\ell^{d-2}. \tag{93}$$

A radius change $\delta\ell$ contributes

$$\delta_\ell V = A_0\delta\ell, \qquad \delta_\ell A = \frac{d-2}{\ell}A_0\delta\ell. \tag{94}$$

Setting the total volume variation to zero gives $A_0\delta\ell = -\delta V|_\ell$, so

$$\delta A|_V = \delta A|_\ell - \frac{d-2}{\ell}\delta V|_\ell \tag{95}$$

$$= -\frac{\Omega_{d-2}\ell^d}{2(d^2-1)}\mathcal{R}. \tag{96}$$

Finally, the Gauss relation at the center in the adapted orthonormal frame is

$$\mathcal{R} = 2G_{ab}u^a u^b. \tag{97}$$

This proves

$$\delta A|_V = -\frac{\Omega_{d-2}\ell^d}{d^2-1}G_{ab}u^a u^b. \tag{98}$$

Linearizing relative to a maximally symmetric reference gives (43).

B Conformal Killing normalization

For the flat diamond centered at $t = r = 0$ with tips at $t = \pm\ell$, define null coordinates $u = t - r$ and $v = t + r$. The spherically symmetric conformal Killing vector preserving the diamond is

$$\zeta = \frac{1}{2\ell}\left[(\ell^2 - u^2)\partial_u + (\ell^2 - v^2)\partial_v\right]. \tag{99}$$

In (t, r) coordinates,

$$\zeta = \frac{1}{2\ell}\left[(\ell^2 - r^2 - t^2)\partial_t - 2rt\partial_r\right]. \tag{100}$$

It vanishes on the edge $t = 0, r = \ell$ and at the tips. Its conformal factor is

$$\mathcal{L}_\zeta \eta_{ab} = -\frac{2t}{\ell}\eta_{ab}, \tag{101}$$

and its conformal Killing horizon has surface gravity $\kappa = 1$ in this normalization. On the central slice, (100) reduces to (44). The modular flow vector is $(2\pi/\hbar)\zeta^a$. This accounts for every factor in (47).

For a boundary ball in the holographic route, the boundary vector is normalized directly as the modular flow generator, so its bulk extension has surface gravity 2π in units $\hbar = 1$. The two conventions are equivalent: one places 2π in the vector, the other in the modular temperature.

C Iyer-Wald sign and orientation audit

Let

$$\delta \mathbf{L} = \mathbf{E} \cdot \delta\phi + \mathrm{d}\boldsymbol{\Theta}(\phi; \delta\phi) \tag{102}$$

for fields ϕ. The Noether current associated with a vector ξ is

$$\boldsymbol{J}_\xi = \boldsymbol{\Theta}(\phi; \mathcal{L}_\xi\phi) - \xi \cdot \mathbf{L}. \tag{103}$$

It decomposes as

$$\boldsymbol{J}_\xi = \mathrm{d}\boldsymbol{Q}_\xi + \boldsymbol{C}_\xi, \tag{104}$$

where $\boldsymbol{C}_\xi$ vanishes on the constraints. On a background for which $\mathcal{L}_\xi\phi = 0$, variation gives

$$\mathrm{d}\left(\delta\boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\Theta}\right) = -\delta\boldsymbol{C}_\xi. \tag{105}$$

For pure metric gravity with the normalization (25),

$$\delta\boldsymbol{C}_\xi = 2\xi^a \delta E_{ab}\boldsymbol{\varepsilon}^b, \tag{106}$$

which gives (27).

Orient Σ_B as in the standard AdS-Rindler first-law convention, so that its boundary is $\widetilde{B} \cup (-B)$. Stokes' theorem reads

$$\int_{\Sigma_B} \mathrm{d}\boldsymbol{\chi}_B = \int_{\widetilde{B}} \boldsymbol{\chi}_B - \int_B \boldsymbol{\chi}_B. \tag{107}$$

The two boundary terms are respectively geometric entropy and modular energy. Hence

$$\delta\langle K_B\rangle - \delta S_B = -\int_{\Sigma_B} \mathrm{d}\boldsymbol{\chi}_B = +2\int_{\Sigma_B} \xi_B^a \delta E_{ab} \boldsymbol{\varepsilon}^b. \tag{108}$$

This sign agrees both with $D = \Delta\langle K\rangle - \Delta S$ and, when $\delta E_{ab} = \frac{1}{2}\delta\langle T_{ab}^{\mathrm{bulk}}\rangle$, with the positive bulk modular energy in the JLMS relation. Reversing the orientation of Σ_B also reverses the induced form $\boldsymbol{\varepsilon}^b$ and changes no physical equation.

If matter is included in the total Lagrangian with

$$\delta\Gamma_{\mathrm{m}} = -\frac{1}{2}\int \sqrt{-g}\, T_{ab}\delta g^{ab}, \tag{109}$$

the total metric Euler-Lagrange tensor is

$$E_{ab}^{\mathrm{tot}} = \frac{1}{16\pi G}(G_{ab} + \Lambda g_{ab}) - \frac{1}{2}T_{ab}. \tag{110}$$

The constraint $E_{ab}^{\mathrm{tot}} = 0$ is therefore exactly $G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$.

D Scope ledger for the central equations

Equation	Domain	What it does not assert
$D = \Delta\langle K\rangle - \Delta S$	Same algebra and fixed faithful reference	Existence of a geometric dual
$\delta S = \delta\langle K\rangle$	Differentiable normalized path through the reference	Positivity at second order or a field equation
Ball modular Hamiltonian (14)	CFT vacuum and ball domain of dependence	Generic QFT, region, or reference state
RT (21)	Leading classical static holographic regime	Bulk quantum entropy or arbitrary nonholographic area laws
Iyer-Wald constraint (31)	Linear perturbation about an AdS-Rindler Killing background	Finite nonlinear solution from one ball
Fixed-volume area (42)	Small geodesic ball to first curvature order	Fixed-radius formula or arbitrary surface
Diamond Einstein relation (54)	Local equilibrium, CFT matter, universal η, all frames	Generic nonconformal matter or a determined Λ
Higher-curvature equilibrium (74)	Maximally symmetric reference, matched entropy and generalized volume	Full nonlinear higher-curvature equations

Equation	Domain	What it does not assert
Canonical energy (81)	Holographic code subspace, on-shell perturbation, controlled gauge	Generic information metric equals gravitational stiffness

References

[1] H. Araki, "Relative entropy of states of von Neumann algebras," Publ. Res. Inst. Math. Sci. Kyoto **11** (1976) 809–833.

[2] J. J. Bisognano and E. H. Wichmann, "On the duality condition for a Hermitian scalar field," J. Math. Phys. **16** (1975) 985–1007; "On the duality condition for quantum fields," J. Math. Phys. **17** (1976) 303–321.

[3] D. D. Blanco, H. Casini, L.-Y. Hung, and R. C. Myers, "Relative entropy and holography," JHEP **08** (2013) 060, arXiv:1305.3182.

[4] P. Bueno, V. S. Min, A. J. Speranza, and M. R. Visser, "Entanglement equilibrium for higher order gravity," Phys. Rev. D **95** (2017) 046003, arXiv:1612.04374.

[5] H. Casini, D. A. Galante, and R. C. Myers, "Comments on Jacobson's entanglement equilibrium and the Einstein equation," JHEP **03** (2016) 194, arXiv:1601.00528.

[6] H. Casini, M. Huerta, and R. C. Myers, "Towards a derivation of holographic entanglement entropy," JHEP **05** (2011) 036, arXiv:1102.0440.

[7] T. Faulkner, M. Guica, T. Hartman, R. C. Myers, and M. Van Raamsdonk, "Gravitation from entanglement in holographic CFTs," JHEP **03** (2014) 051, arXiv:1312.7856.

[8] T. Faulkner, A. Lewkowycz, and J. Maldacena, "Quantum corrections to holographic entanglement entropy," JHEP **11** (2013) 074, arXiv:1307.2892.

[9] S. Hollands and R. M. Wald, "Stability of black holes and black branes," Commun. Math. Phys. **321** (2013) 629–680, arXiv:1201.0463.

[10] V. E. Hubeny, M. Rangamani, and T. Takayanagi, "A covariant holographic entanglement entropy proposal," JHEP **07** (2007) 062, arXiv:0705.0016.

[11] V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy," Phys. Rev. D **50** (1994) 846–864, arXiv:gr-qc/9403028.

[12] T. Jacobson, "Thermodynamics of spacetime: The Einstein equation of state," Phys. Rev. Lett. **75** (1995) 1260–1263, arXiv:gr-qc/9504004.

[13] T. Jacobson, "Entanglement equilibrium and the Einstein equation," Phys. Rev. Lett. **116** (2016) 201101, arXiv:1505.04753.

[14] D. L. Jafferis, A. Lewkowycz, J. Maldacena, and S. J. Suh, "Relative entropy equals bulk relative entropy," JHEP **06** (2016) 004, arXiv:1512.06431.

[15] N. Lashkari, M. B. McDermott, and M. Van Raamsdonk, "Gravitational dynamics from entanglement thermodynamics," JHEP **04** (2014) 195, arXiv:1308.3716.

[16] N. Lashkari and M. Van Raamsdonk, "Canonical energy is quantum Fisher information," JHEP **04** (2016) 153, arXiv:1508.00897.

[17] S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from AdS/CFT," Phys. Rev. Lett. **96** (2006) 181602, arXiv:hep-th/0603001.

[18] A. J. Speranza, "Entanglement entropy of excited states in conformal perturbation theory and the Einstein equation," JHEP **04** (2016) 105, arXiv:1602.01380.

[19] R. M. Wald, "Black hole entropy is the Noether charge," Phys. Rev. D **48** (1993) R3427–R3431, arXiv:gr-qc/9307038.