The Infrared Consistency Equation for Reconstructed Metrics

Matthew Long
YonedaAI Research Collective
Chicago, Illinois, USA
matthew@yonedaai.com

August 3, 2026

Abstract

Suppose that coarse-grained quantum data admit a smooth Lorentzian metric reconstruction and that this metric is a field of a local infrared effective theory. This paper identifies the additional hypotheses under which stationarity of that theory gives the Einstein equation at leading derivative order. We vary an action containing the Einstein-Hilbert term, a cosmological term, matter, and representative curvature-squared operators. The stress tensor convention and every factor of $8\pi G_{\mathrm{N}}$ are fixed explicitly. The Gibbons-Hawking-York term is included for a non-null Dirichlet boundary, while null boundaries, corners, and higher-curvature boundary data are kept as separate qualifications. The central observation is functional rather than algebraic: stationarity with respect to microscopic reconstruction parameters implies the full metric Euler-Lagrange equation only when admissible reconstruction variations span the local metric variations modulo diffeomorphisms. Otherwise one obtains only a projection of that equation. Under locality, diffeomorphism redundancy, a derivative expansion, a single interacting massless spin-2 metric mode, no other unsuppressed long-range fields, and controlled Wilson coefficients, the projected equation reduces to the Einstein equation with local higher-derivative and nonlocal massless-loop corrections. Dimension-dependent Lovelock terms and the special cases of two and three spacetime dimensions are treated separately. The result is a conditional infrared classification statement, not a construction of a metric from an arbitrary quantum system.

Contents

1 Introduction

A metric obtained from coarse-grained information is not governed by the Einstein equation merely because it is called a spacetime metric. Three logically distinct questions intervene. First, does the microscopic system admit a smooth reconstruction whose output contains a Lorentzian metric? Second, is that metric a genuine variational field of a diffeomorphism- redundant infrared theory? Third, which terms dominate the metric equation at the length scale where the reconstruction is used? Only the second and third questions are addressed here. The existence of the reconstruction is an input.

The low-energy action method gives a precise answer to the third question. If the surviving gravitational spectrum contains one massless spin-2 field represented by a metric, local diffeomorphism-invariant operators can be organized by their number of derivatives. The cosmological operator has no derivatives, the Einstein-Hilbert operator has two, and curvature-squared operators have four. Massive degrees of freedom contribute local operators below their thresholds, while loops of massless fields also produce nonlocal, nonanalytic form factors. This is the standard effective-field-theory organization of gravity [13, 16].

That organization is sometimes compressed into the assertion that general relativity is automatic in the infrared. The assertion is too strong without spectral and coefficient assumptions. A scalar-tensor theory contains an additional long-range scalar. A theory with several interacting spin-2 fields has a different field space. A large coefficient multiplying a four-derivative operator can defeat naive momentum suppression. In dimensions greater than four, Lovelock densities provide dimension-sensitive exceptions to a classification stated only in terms of differential order. In two dimensions the Einstein-Hilbert integral is topological, and in three dimensions pure Einstein gravity has no local graviton polarization.

The second question contains a separate subtlety. Let z denote microscopic or mesoscopic reconstruction data, let $g = \mathfrak{R}(z)$, and consider the pulled-back functional $\Gamma_{\mathrm{IR}}[\mathfrak{R}(z), \Phi(z)]$. Its derivative is the adjoint of the linearized reconstruction map applied to the field equations. Thus stationarity in z gives

$$(D\mathfrak{R}_z)^*\mathcal{C} = 0, \tag{1}$$

not automatically $\mathcal{C} = 0$. The latter follows only if the allowed image of $D\mathfrak{R}_z$ is sufficiently rich after gauge directions and boundary conditions are accounted for. A reconstruction that permits only conformal variations, for example, probes only the trace of the metric equation.

The aim of this paper is to derive the tensor $\mathcal{C}_{\mu\nu}$, state a clean condition under which equation (1) implies its vanishing, and separate the local infrared approximation from its known qualifications. The main equation will be

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + H^{\mathrm{loc}}_{\mu\nu} + H^{\mathrm{nl}}_{\mu\nu} = 8\pi G_{\mathrm{N}} \langle T_{\mu\nu}\rangle, \tag{2}$$

with every correction tensor defined by a functional derivative. At two-derivative order, and under the hypotheses stated below, this becomes

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_{\mathrm{N}} \langle T_{\mu\nu}\rangle. \tag{3}$$

The ingredients in this derivation are established. The variational structure and its Noether identity go back to the action formulation of general relativity and to Noether's theorems [1, 14]. The boundary completion is due to York and Gibbons and Hawking [5, 6]. The spin-2 consistency argument is represented by the classic analyses of Fierz and Pauli, Weinberg, and Deser [2, 3, 4]. Lovelock's theorem classifies a specified class of natural divergence-free tensors [7, 8]. The contribution made here is to place these ingredients behind an explicit reconstruction-variation interface and to show exactly what stationarity does and does not imply at that interface.

2 Geometric and variational conventions

Let M be a smooth, oriented, time-oriented manifold of spacetime dimension $d \geq 2$. The metric signature is

$$(-,+,\dots,+). \tag{4}$$

The Levi-Civita connection is denoted by ∇. Curvature conventions are fixed by

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma, \tag{5}$$

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \tag{6}$$

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{7}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{8}$$

We write $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. All bulk metric variations are taken with respect to the inverse metric $g^{\mu\nu}$, so

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}. \tag{9}$$

The renormalized matter effective action is $\Gamma_{\mathrm{m}}[g,\Phi]$, where Φ collects matter fields, sources, and any infrared expectation values retained in the description. Our stress-tensor convention is

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta\Gamma_{\mathrm{m}}}{\delta g^{\mu\nu}}. \tag{10}$$

Consequently,

$$\delta_g\Gamma_{\mathrm{m}} = -\frac{1}{2}\int_M \mathrm{d}^d x\,\sqrt{-g}\,T_{\mu\nu}\delta g^{\mu\nu}, \tag{11}$$

up to boundary variations and possible variations of external sources. In a semiclassical equation, $T_{\mu\nu}$ in equation (10) is the renormalized expectation value determined by the matter effective action.

Definition 2.1 (Normalized metric Euler-Lagrange tensor)**.** For the gravitational part Γ_{g} of the effective action, set

$$\mathcal{E}_{\mu\nu}[g,\Phi] := \frac{16\pi G_{\mathrm{N}}}{\sqrt{-g}}\frac{\delta\Gamma_{\mathrm{g}}}{\delta g^{\mu\nu}}. \tag{12}$$

The normalization is chosen so that the Einstein-Hilbert contribution is $G_{\mu\nu} + \Lambda g_{\mu\nu}$.

Definition 2.2 (Metric consistency tensor)**.** The metric consistency tensor is

$$\mathcal{C}_{\mu\nu}[g,\Phi] := \mathcal{E}_{\mu\nu}[g,\Phi] - 8\pi G_{\mathrm{N}} T_{\mu\nu}. \tag{13}$$

The full metric equation is $\mathcal{C}_{\mu\nu} = 0$.

The factors in equations (12) and (13) follow directly from equation (11). Indeed, if

$$\delta_g \Gamma_{\mathrm{g}} = \frac{1}{16\pi G_{\mathrm{N}}} \int_M \mathrm{d}^d x \sqrt{-g}\, \mathcal{E}_{\mu\nu} \delta g^{\mu\nu}, \tag{14}$$

then $\delta_g(\Gamma_{\mathrm{g}} + \Gamma_{\mathrm{m}}) = 0$ for arbitrary admissible metric variations gives

$$\frac{1}{16\pi G_{\mathrm{N}}} \mathcal{E}_{\mu\nu} - \frac{1}{2} T_{\mu\nu} = 0, \tag{15}$$

which is equivalent to equation (13).

3 The reconstruction interface

The microscopic origin of a reconstruction need not be fixed in order to state the variational issue. Let $\mathcal{Q}$ denote a space of microscopic or mesoscopic data. Its points may include a local algebra net, a state, a coarse-graining prescription, and a specified information metric. At a scale μ, a reconstruction is a map

$$\mathfrak{R}_\mu : \mathcal{Q} \longrightarrow \mathcal{F}_{\mathrm{IR}}, \qquad q \longmapsto (M, g(q), \Phi(q); \boldsymbol{c}(\mu)), \tag{16}$$

where $\mathcal{F}_{\mathrm{IR}}$ is the infrared field space and $\boldsymbol{c}(\mu)$ is its set of effective couplings. This paper assumes that the metric component of equation (16) is smooth in the neighborhood under study.

Let $T_q\mathcal{Q}$ be the space of admissible reconstruction perturbations at q. Boundary conditions, reality conditions, locality restrictions, constraints, and the chosen state family are part of the word "admissible." Linearizing the metric component gives

$$D\mathfrak{R}^{(g)}_{\mu,q} : T_q\mathcal{Q} \longrightarrow T_g \operatorname{Met}(M), \qquad \delta q \longmapsto \delta_q g_{\mu\nu}. \tag{17}$$

Infinitesimal diffeomorphisms have the form

$$\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}. \tag{18}$$

They are redundant directions rather than independent physical metric polarizations.

Definition 3.1 (Admissible metric variation space)**.** Fix a boundary condition B and a regularity class. Let

$$\mathcal{V}_{g,B} \subset \Gamma(S^2 T^* M) \tag{19}$$

be the linear space of local metric variations preserving B. Its gauge subspace is

$$\mathcal{V}^{\mathrm{gauge}}_{g,B} = \{\mathcal{L}_\xi g : \xi \text{ preserves } B\}. \tag{20}$$

The physical variation space is the quotient $\mathcal{V}_{g,B}/\mathcal{V}^{\mathrm{gauge}}_{g,B}$.

Assumption 3.2 (Reconstruction-tangent completeness)**.** At the background q, the image of $D\mathfrak{R}^{(g)}_{\mu,q}$ is dense in $\mathcal{V}_{g,B}$ in a topology for which pairing with the Euler-Lagrange tensor is continuous, after adjoining gauge variations if necessary. In a finite-order local argument, it is enough to require that every compactly supported smooth symmetric tensor can be represented, modulo a Lie derivative, by an admissible reconstruction perturbation.

The density language covers reconstructions described by infinitely many coordinates. For a finite-dimensional reconstruction ansatz, the assumption usually fails: a finite-dimensional tangent space cannot span arbitrary local metric variations. Such an ansatz can consistently test selected components or moments of the field equation, but not the complete tensor equation.

Assumption 3.3 (Independent matter variations)**.** The matter equations have been imposed, or the admissible perturbations allow metric and matter variations to be separated at first order. Thus a metric variation of the pulled-back effective action can be evaluated with the matter Euler-Lagrange terms vanishing.

This assumption prevents a cancellation between a nonzero metric equation and a correlated off-shell matter variation. It may be replaced by a more general statement that the combined reconstruction tangent is complete in the full field space. The simpler form is sufficient for the metric result.

4 The local infrared action

At scales well below a massive threshold M_*, the local part of a diffeomorphism-invariant metric effective action can be organized by derivative order. To quadratic order in curvature, write

$$\begin{aligned}\Gamma_{\mathrm{loc}}[g] = {}& \frac{1}{16\pi G_{\mathrm{N}}}\int_M \mathrm{d}^d x\,\sqrt{-g}\,(R-2\Lambda)\\ &+\int_M \mathrm{d}^d x\,\sqrt{-g}\,\left[c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right]\\ &+\Gamma_{\geq 6}[g].\end{aligned} \tag{21}$$

Here $\Gamma_{\geq 6}$ contains local operators with at least six derivatives, including curvature-cubed and derivative-of-curvature terms. The couplings c_i are renormalized Wilson coefficients at the working scale. In four dimensions they are dimensionless in units with $\hbar = c = 1$; in general their mass dimensions depend on d.

The complete action used below is

$$\Gamma_{\mathrm{IR}}[g,\Phi] = \Gamma_{\mathrm{loc}}[g] + \Gamma_{\mathrm{nl}}[g,\Phi] + \Gamma_{\mathrm{m}}[g,\Phi], \tag{22}$$

where Γ_{nl} contains nonlocal terms, especially those induced by massless loops. The split between Γ_{loc} and Γ_{nl} depends on renormalization scheme, but nonanalytic momentum dependence cannot be absorbed into a finite set of local coefficients.

Assumption 4.1 (Infrared field content)**.** The gravitational sector contains one interacting massless spin-2 mode whose gauge redundancy is represented by diffeomorphisms of a single metric. No additional massless scalar, vector, tensor, torsion, or nonmetricity field couples with unsuppressed strength in the regime of interest.

The assumption is stronger than the statement that the action is written with a metric. Higher-derivative terms treated nonperturbatively can introduce additional poles, as in quadratic gravity [9]. In an effective theory these terms are instead expanded perturbatively below the cutoff unless the corresponding new state is intentionally retained. The classic self-coupling arguments show how a consistent interacting massless spin-2 field leads to the nonlinear gauge structure of general relativity under locality and universal coupling assumptions [3, 4]. They do not prove that a given microscopic system contains such a field, and they do not exclude a larger massless spectrum.

Assumption 4.2 (Controlled derivative expansion)**.** There is a length $\ell_* = M_*^{-1}$ and a background curvature scale L such that $\ell_*/L \ll 1$. Dimensionless combinations of Wilson coefficients are not parametrically large enough to compensate for the associated powers of ℓ_*/L. Curvature and its derivatives vary on scales of order L.

This coefficient condition is essential. Derivative counting says that R^2 has two more derivatives than R, but the ratio in the equation of motion is actually of the form

$$\frac{16\pi G_{\mathrm{N}} c_i H^{(i)}}{G} \sim \widehat{c}_i \frac{\ell_*^2}{L^2}, \tag{23}$$

with a convention-dependent dimensionless coefficient $\widehat{c}_i$. If $\widehat{c}_i(\ell_*^2/L^2)$ is not small, the four-derivative term is not a controlled correction.

The cosmological operator is different. It has fewer derivatives than the Einstein-Hilbert term and is not suppressed at long distance. A background with curvature radius L_Λ satisfies $|\Lambda| \sim L_\Lambda^{-2}$. The cosmological term belongs in the leading equation even when its small observed value requires an explanation outside the derivative expansion.

5 Einstein-Hilbert variation and boundary data

The bulk Einstein-Hilbert variation follows from the Palatini identity

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\nu \delta\Gamma^\rho_{\mu\rho}. \tag{24}$$

For an inverse-metric variation one obtains

$$\delta(\sqrt{-g}R) = \sqrt{-g}\, G_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\,\nabla_\mu v^\mu, \tag{25}$$

where one convenient expression for the boundary vector is

$$v^\mu = g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\beta_{\alpha\beta}. \tag{26}$$

The cosmological term varies as

$$\delta\left[-2\Lambda\sqrt{-g}\right] = \sqrt{-g}\,\Lambda g_{\mu\nu}\delta g^{\mu\nu}. \tag{27}$$

If M is closed, or if variations have compact support in its interior, the divergence in equation (25) does not contribute. For a smooth non-null boundary ∂M with induced metric h_{ab}, unit normal n^μ, and $n^\mu n_\mu = \varepsilon = \pm 1$, the Dirichlet action is

$$\begin{aligned}\Gamma_{\mathrm{EH+D}}[g] = {}& \frac{1}{16\pi G_{\mathrm{N}}}\int_M \mathrm{d}^d x\sqrt{-g}\,(R-2\Lambda) \\ & + \frac{\varepsilon}{8\pi G_{\mathrm{N}}}\int_{\partial M}\mathrm{d}^{d-1}y\sqrt{|h|}\,K,\end{aligned} \tag{28}$$

where $K = h^{\mu\nu}\nabla_\mu n_\nu$. With the boundary orientation and extrinsic-curvature convention fixed as above, the second term cancels normal derivatives of $\delta g_{\mu\nu}$ when $\delta h_{ab} = 0$ [5, 6].

The variation then has the form

$$\begin{aligned}\delta\Gamma_{\mathrm{EH+D}} = {}& \frac{1}{16\pi G_{\mathrm{N}}}\int_M \mathrm{d}^d x\sqrt{-g}\,(G_{\mu\nu} + \Lambda g_{\mu\nu})\delta g^{\mu\nu} \\ & - \frac{\varepsilon}{16\pi G_{\mathrm{N}}}\int_{\partial M}\mathrm{d}^{d-1}y\sqrt{|h|}\,(K_{ab} - K h_{ab})\delta h^{ab}.\end{aligned} \tag{29}$$

The boundary integral vanishes for Dirichlet data. Equation (29) fixes the sign and normalization of the bulk Einstein tensor used throughout the paper.

Several qualifications should not be hidden inside the phrase "add the boundary term." Null boundaries require a different treatment because the induced metric is degenerate. Piecewise smooth boundaries require joint or corner terms. Asymptotically flat and asymptotically anti-de Sitter problems also require falloff conditions and, in the latter case, counterterms. For curvature-squared actions, the Gibbons-Hawking-York term alone does not define a Dirichlet variational principle. One must specify additional boundary data or add the boundary completion appropriate to the chosen higher-curvature theory. The bulk equations below are therefore derived using compactly supported variations, a closed manifold, or a boundary problem already completed so that its residual boundary variation vanishes.

Proposition 5.1 (Einstein-Hilbert contribution)**.** *Under any of the boundary conditions just stated, the normalized Euler-Lagrange tensor of equation (28) is*

$$\mathcal{E}^{(2)}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{30}$$

With the stress convention equation (10), stationarity of $\Gamma_{\mathrm{EH+D}} + \Gamma_{\mathrm{m}}$ *under arbitrary admissible metric variations is equivalent to*

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}. \tag{31}$$

Proof. The bulk part of equation (29) and the matter variation equation (11) give

$$\delta_g(\Gamma_{\mathrm{EH+D}} + \Gamma_{\mathrm{m}}) = \int_M \mathrm{d}^d x\sqrt{-g}\left[\frac{G_{\mu\nu} + \Lambda g_{\mu\nu}}{16\pi G_{\mathrm{N}}} - \frac{1}{2}T_{\mu\nu}\right]\delta g^{\mu\nu}. \tag{32}$$

The fundamental lemma of the calculus of variations gives the bracketed coefficient as zero. Multiplication by $16\pi G_{\mathrm{N}}$ yields equation (31). □

6 Representative curvature-squared variations

Define tensors $H^{(i)}_{\mu\nu}$ by

$$\delta\int_M \mathrm{d}^d x\sqrt{-g}\,R^2 = \int_M \mathrm{d}^d x\sqrt{-g}\,H^{(1)}_{\mu\nu}\delta g^{\mu\nu}, \tag{33}$$

$$\delta\int_M \mathrm{d}^d x\sqrt{-g}\,R_{\alpha\beta}R^{\alpha\beta} = \int_M \mathrm{d}^d x\sqrt{-g}\,H^{(2)}_{\mu\nu}\delta g^{\mu\nu}, \tag{34}$$

$$\delta\int_M \mathrm{d}^d x\sqrt{-g}\,R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \int_M \mathrm{d}^d x\sqrt{-g}\,H^{(3)}_{\mu\nu}\delta g^{\mu\nu}, \tag{35}$$

after the relevant boundary contribution has been removed. With the curvature conventions of section 2, direct variation gives

$$H^{(1)}_{\mu\nu} = 2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2 + 2(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R, \tag{36}$$

$$\begin{aligned} H^{(2)}_{\mu\nu} &= 2R_{\mu\alpha\nu\beta}R^{\alpha\beta} - \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \Box R_{\mu\nu} \\ &\quad + \frac{1}{2}g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R. \end{aligned} \tag{37}$$

These formulas display the fourth derivatives that enter a generic quadratic metric equation.

It is useful to introduce the quadratic Euler density

$$\mathcal{X}_4 := R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \tag{38}$$

Its bulk variation defines the Lanczos tensor

$$\begin{aligned} H^{(\mathrm{GB})}_{\mu\nu} = 2\big(&RR_{\mu\nu} - 2R_{\mu\alpha}R_\nu{}^\alpha - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} \\ &+ R_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}\big) - \frac{1}{2}g_{\mu\nu}\mathcal{X}_4. \end{aligned} \tag{39}$$

Linearity of the variation gives the identity

$$H^{(3)}_{\mu\nu} = H^{(\mathrm{GB})}_{\mu\nu} + 4H^{(2)}_{\mu\nu} - H^{(1)}_{\mu\nu}. \tag{40}$$

This relation is often safer than using an isolated Riemann-squared formula, because it makes the four-dimensional Euler cancellation explicit.

Each $H^{(i)}_{\mu\nu}$ is covariantly conserved as a local identity,

$$\nabla^\mu H^{(i)}_{\mu\nu} = 0, \tag{41}$$

provided the corresponding scalar action is diffeomorphism invariant. This is a Noether identity and does not require the metric equation. It is also a useful sign check on equations (36) and (37).

For the local action equation (21), define

$$H^{\mathrm{loc}}_{\mu\nu} := 16\pi G_{\mathrm{N}}\left(c_1 H^{(1)}_{\mu\nu} + c_2 H^{(2)}_{\mu\nu} + c_3 H^{(3)}_{\mu\nu} + H^{(\geq 6)}_{\mu\nu}\right), \tag{42}$$

where

$$H^{(\geq 6)}_{\mu\nu} := \frac{1}{\sqrt{-g}}\frac{\delta\Gamma_{\geq 6}}{\delta g^{\mu\nu}}. \tag{43}$$

The factor $16\pi G_{\mathrm{N}}$ in equation (42) is not optional: the c_i terms in equation (21) were written outside the $1/(16\pi G_{\mathrm{N}})$ prefactor.

Proposition 6.1 (Local metric equation through four derivatives)**.** *For the action $\Gamma_{\mathrm{loc}} + \Gamma_{\mathrm{m}}$, subject to a well-posed boundary variation, the metric equation is*

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + 16\pi G_{\mathrm{N}}\left(c_1 H^{(1)}_{\mu\nu} + c_2 H^{(2)}_{\mu\nu} + c_3 H^{(3)}_{\mu\nu}\right) + \mathcal{O}(\partial^6) = 8\pi G_{\mathrm{N}} T_{\mu\nu}. \tag{44}$$

Proof. Combine equations (30) and (33) to (35) with equation (11). The gravitational variation is

$$\delta_g \Gamma_{\text{loc}} = \int_M \mathrm{d}^d x \sqrt{-g} \left[\frac{G_{\mu\nu} + \Lambda g_{\mu\nu}}{16\pi G_{\mathrm{N}}} + c_1 H^{(1)}_{\mu\nu} + c_2 H^{(2)}_{\mu\nu} + c_3 H^{(3)}_{\mu\nu} + H^{(\geq 6)}_{\mu\nu} \right] \delta g^{\mu\nu}. \tag{45}$$

Subtracting $T_{\mu\nu}/2$ from the matter variation and multiplying the Euler-Lagrange coefficient by $16\pi G_{\mathrm{N}}$ gives equation (44). □

Two internal checks are immediate. In $d = 4$, the traces of equations (36) and (37) are

$$g^{\mu\nu} H^{(1)}_{\mu\nu} = 6\Box R, \tag{46}$$

$$g^{\mu\nu} H^{(2)}_{\mu\nu} = 2\Box R. \tag{47}$$

The trace of the four-dimensional Gauss-Bonnet variation vanishes. On a four-dimensional Einstein metric $R_{\mu\nu} = \lambda g_{\mu\nu}$ with constant λ, both $H^{(1)}_{\mu\nu}$ and $H^{(2)}_{\mu\nu}$ vanish. Thus every four-dimensional Einstein metric remains a solution of the vacuum quadratic terms at first order in their coefficients, although more general solutions and perturbative corrections can occur.

7 A general local curvature functional

The preceding calculation can be packaged in a form useful beyond a selected quadratic basis. Let

$$I_F[g] = \int_M \mathrm{d}^d x \sqrt{-g}\, F(g^{\mu\nu}, R_{\mu\nu\rho\sigma}), \tag{48}$$

where F is a scalar constructed algebraically from the metric and Riemann tensor, without explicit covariant derivatives of curvature. Define

$$P^{\mu\nu\rho\sigma} := \frac{\partial F}{\partial R_{\mu\nu\rho\sigma}}. \tag{49}$$

This derivative is taken with the metric held fixed and inherits the algebraic symmetries of the Riemann tensor. After the boundary variation is removed, the bulk Euler-Lagrange tensor is

$$E^{(F)}_{\mu\nu} = P_\mu{}^{\alpha\beta\gamma} R_{\nu\alpha\beta\gamma} - 2\nabla^\alpha \nabla^\beta P_{\mu\alpha\beta\nu} - \frac{1}{2} g_{\mu\nu} F. \tag{50}$$

The first term is symmetric after the curvature symmetries and scalar nature of F are used. The second term contains the higher derivatives. For a generic polynomial of degree k in curvature, P has degree $k - 1$, and its double derivative produces derivatives of the metric above second order.

For $F = R$, one has

$$P^{\mu\nu\rho\sigma}_R = \frac{1}{2} \left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} \right). \tag{51}$$

Metric compatibility makes the double derivative vanish, and equation (50) reduces to $G_{\mu\nu}$. For $F = R^2$,

$$P^{\mu\nu\rho\sigma}_{R^2} = R \left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} \right), \tag{52}$$

and equation (50) reduces to equation (36). These substitutions independently verify both the Einstein-Hilbert sign and the relative sign of the R^2 derivative terms.

The Lovelock densities are distinguished by

$$\nabla_\mu P^{\mu\nu\rho\sigma}_{(k)} = 0. \tag{53}$$

For them the double-derivative term in equation (50) vanishes identically, leaving field equations with no derivatives of the metric above second order. This is the mechanism behind the result in section 12. It does not reduce curvature order: the kth Lovelock tensor still scales as L^{-2k} on a background of curvature radius L.

If F contains derivatives of curvature, equation (50) must be extended by further integrations by parts. For example,

$$\int \sqrt{-g}\, R\Box R = -\int \sqrt{-g}\,(\nabla R)^2 \tag{54}$$

up to a boundary term. Either form produces a six-derivative metric tensor. The equivalence requires compatible boundary data, so local integration by parts cannot be separated from the qualifications in section 5.

Proposition 7.1 (Noether identity for a local curvature scalar)**.** *For any diffeomorphism-invariant I_F with a well-posed boundary variation,*

$$\nabla^\mu E^{(F)}_{\mu\nu} = 0. \tag{55}$$

Proof. Under a compactly supported diffeomorphism, $\delta g^{\mu\nu} = -2\nabla^{(\mu}\xi^{\nu)}$. Invariance and one integration by parts give

$$0 = \delta_\xi I_F = 2\int_M \mathrm{d}^d x\sqrt{-g}\,(\nabla^\mu E^{(F)}_{\mu\nu})\xi^\nu. \tag{56}$$

Since ξ^ν is arbitrary, the divergence vanishes. The argument also applies to a local scalar with curvature derivatives when its complete Euler-Lagrange tensor is used. □

8 Light fields and the operator basis

The split in equation (22) between gravitational and matter pieces is partly conventional when light fields couple to curvature. Consider

$$\Gamma_\mathrm{m}[g,\phi] = -\frac{1}{2}\int_M \mathrm{d}^d x\sqrt{-g}\left[(\nabla\phi)^2 + m^2\phi^2 + \xi R\phi^2\right]. \tag{57}$$

The $\xi R\phi^2$ operator may remain in the matter action and contribute to the stress tensor in equation (10), or it may be moved to a field-dependent coefficient of R. The descriptions agree when all variations are performed consistently. They disagree if the scalar equation or the associated boundary variation is omitted.

If $mL \gg 1$, integrating out ϕ produces local operators organized in powers of $1/m$. If it is massless or light on scale L, it remains in the infrared field space and can generate nonlocal form factors through loops. A light expectation value can also make the effective coefficient of R spacetime dependent. The one-massless-spin-2 assumption does not exclude ordinary massless matter, but it requires its stress tensor and loop effects to be retained.

Gauge fields and fermions generate mixed operators such as

$$RF_{\mu\nu}F^{\mu\nu}, \qquad R_{\mu\nu}F^{\mu\alpha}F^\nu{}_\alpha, \qquad R\,\bar\psi\psi. \tag{58}$$

These terms correct both the metric and matter equations. A pure-metric basis is sufficient for displaying the representative tensors in section 6, but it is not the most general infrared action in a matter background.

The renormalized stress tensor also contains local scheme dependence. A finite curvature counterterm may be assigned to the gravitational action, or its variation may be regarded as a local shift of the stress tensor. The separate coefficients and $\langle T_{\mu\nu}\rangle$ change under this bookkeeping choice, but the complete consistency tensor does not.

Lemma 8.1 (Counterterm invariance)**.** *Let $I_\mathrm{ct}[g]$ be a finite diffeomorphism-invariant local counterterm. Under*

$$\Gamma_\mathrm{g} \mapsto \Gamma_\mathrm{g} + I_\mathrm{ct}, \qquad \Gamma_\mathrm{m} \mapsto \Gamma_\mathrm{m} - I_\mathrm{ct}, \tag{59}$$

the total action and $\mathcal{C}_{\mu\nu}$ are unchanged.

Proof. The total action is unchanged. The shift of $\mathcal{E}_{\mu\nu}$ is $16\pi G_\mathrm{N}(\sqrt{-g})^{-1}\delta I_\mathrm{ct}/\delta g^{\mu\nu}$. The stress tensor shifts by $2(\sqrt{-g})^{-1}\delta I_\mathrm{ct}/\delta g^{\mu\nu}$ because of the minus sign in equation (10). Multiplication by $8\pi G_\mathrm{N}$ gives the same tensor, so the shifts cancel in $\mathcal{C}$. □

The consistency equation must therefore be formulated with the complete renormalized functional, not an arbitrary split into geometry and matter. The measured Newton coupling is fixed by a renormalization condition. Once that condition is chosen, the normalization in equation (31) is unambiguous.

9 Stationarity along reconstruction variations

Let

$$\widehat{\Gamma}_\mu[q] := \Gamma_{\mathrm{IR}}[\mathfrak{R}_\mu(q)] \tag{60}$$

be the effective action pulled back to reconstruction space. For notational simplicity, suppose the matter equations hold. The chain rule gives

$$\delta_q \widehat{\Gamma}_\mu = \frac{1}{16\pi G_{\mathrm{N}}} \int_M \mathrm{d}^d x \sqrt{-g}\, \mathcal{C}_{\mu\nu} \delta_q g^{\mu\nu}. \tag{61}$$

This equation contains the precise meaning of reconstruction stationarity.

Choose a nondegenerate pairing on variations and dual tensor densities,

$$\langle A, h\rangle_g := \int_M \mathrm{d}^d x \sqrt{-g}\, A_{\mu\nu} h^{\mu\nu}. \tag{62}$$

Then equation (61) can be written as

$$D\widehat{\Gamma}_{\mu,q} = \frac{1}{16\pi G_{\mathrm{N}}} (D\mathfrak{R}^{(g)}_{\mu,q})^* \mathcal{C}. \tag{63}$$

Theorem 9.1 (Reconstruction stationarity)**.** *Assume that the effective action has a well-posed first variation, that the matter equations or theorem 3.3 hold, and that theorem 3.2 holds. If*

$$D\widehat{\Gamma}_{\mu,q}[\delta q] = 0 \qquad \textit{for every } \delta q \in T_q\mathcal{Q}, \tag{64}$$

then

$$\mathcal{C}_{\mu\nu}[\mathfrak{R}_\mu(q)] = 0 \tag{65}$$

as a distribution, and hence pointwise when the fields are smooth.

Proof. By equation (61), stationarity implies

$$\langle \mathcal{C}, h\rangle_g = 0 \tag{66}$$

for every h in the image of $D\mathfrak{R}^{(g)}_{\mu,q}$. Gauge directions do not alter the conclusion because diffeomorphism invariance makes the pairing with $\mathcal{L}_\xi g$ vanish, subject to the stated boundary conditions. Tangent completeness and continuity extend the vanishing pairing to all compactly supported admissible symmetric variations. The fundamental lemma for tensor distributions then gives $\mathcal{C}_{\mu\nu} = 0$. ☐

Proposition 9.2 (What follows without tangent completeness)**.** *Without theorem 3.2, stationarity of* $\widehat{\Gamma}_\mu$ *implies only*

$$(D\mathfrak{R}^{(g)}_{\mu,q})^* \mathcal{C} = 0. \tag{67}$$

Equivalently, $\mathcal{C}$ *lies in the annihilator of* $\mathrm{Ran}(D\mathfrak{R}^{(g)}_{\mu,q})$.

Proof. This is equation (63) with $D\widehat{\Gamma}_{\mu,q} = 0$. No inference from a vanishing adjoint image to a vanishing vector is valid unless the original map has a sufficiently large range. ☐

Example 9.3 (Conformal reconstruction)**.** Suppose the reconstructed metrics have the form

$$g_{\mu\nu}(\sigma) = e^{2\sigma} \bar{g}_{\mu\nu} \tag{68}$$

for a fixed $\bar{g}$. Then

$$\delta g^{\mu\nu} = -2\delta\sigma\, g^{\mu\nu}. \tag{69}$$

Equation (61) becomes

$$\delta_\sigma \widehat{\Gamma} = -\frac{1}{8\pi G_{\mathrm{N}}} \int_M \mathrm{d}^d x \sqrt{-g}\, \mathcal{C}^\mu{}_\mu \,\delta\sigma. \tag{70}$$

Stationarity for arbitrary $\delta\sigma$ yields only $\mathcal{C}^\mu{}_\mu = 0$. The traceless part of the metric equation remains undetermined.

Example 9.4 (Homogeneous and isotropic reconstruction)**.** If the reconstructed metric is restricted to a lapse and one scale factor, variation gives the minisuperspace Hamiltonian and evolution equations. These are the homogeneous and isotropic projections of the covariant metric equation. They do not establish the field equation for inhomogeneous or anisotropic perturbations. Substituting an ansatz into an action can also lose an equation if gauge fixing is performed before variation without retaining the corresponding constraint.

Remark 9.5. Tangent completeness is a local identifiability condition. It says neither that every metric arises globally from the reconstruction nor that the microscopic description is unique. It says only that the permitted first-order microscopic changes are capable of testing every physical local metric direction relevant to the Euler-Lagrange equation.

10 The infrared consistency equation

Define the nonlocal correction tensor by

$$H^{\mathrm{nl}}_{\mu\nu} := \frac{16\pi G_{\mathrm{N}}}{\sqrt{-g}} \frac{\delta \Gamma_{\mathrm{nl}}}{\delta g^{\mu\nu}}. \tag{71}$$

Combining the local and nonlocal variations yields

$$\mathcal{C}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} + H^{\mathrm{loc}}_{\mu\nu} + H^{\mathrm{nl}}_{\mu\nu} - 8\pi G_{\mathrm{N}} T_{\mu\nu}. \tag{72}$$

Theorem 10.1 (Infrared consistency equation)**.** *Let a reconstruction equation (16) satisfy the following hypotheses in a neighborhood of a background:*

(i) it supplies a smooth nondegenerate Lorentzian metric and infrared matter fields;

(ii) reconstruction-coordinate redundancy acts as diffeomorphism redundancy on the metric;

(iii) the effective action is diffeomorphism invariant and has a well-posed variational principle for the stated boundary data;

(iv) the matter equations hold and the stress tensor is defined by equation (10);

(v) reconstruction-tangent completeness, theorem 3.2, holds;

(vi) the infrared spectrum and derivative expansion satisfy theorems 4.1 and 4.2;

(vii) there is no uncancelled diffeomorphism anomaly.

If the pulled-back action is stationary for all admissible reconstruction variations, then the reconstructed metric obeys

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + H^{\mathrm{loc}}_{\mu\nu} + H^{\mathrm{nl}}_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}. \tag{73}$$

At leading local two-derivative order,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}, \tag{74}$$

with relative local corrections controlled by powers of ℓ_*^2/L^2 and with nonlocal massless-loop corrections retained whenever their form factors are relevant.

Proof. Hypotheses (i)–(v) allow application of theorem 9.1, so the full normalized tensor equation (72) vanishes. This gives equation (73). Hypothesis (vi) organizes the local terms by derivative order. The unique two-derivative metric kinetic operator within the assumed one-metric massless-spin-2 field content is the Einstein-Hilbert operator, up to its normalization and field redefinitions. The zero-derivative cosmological operator remains at leading order. Controlled four- and higher-derivative Wilson coefficients are suppressed at $L \gg \ell_*$. Massless nonlocal terms are not discarded by this local power-counting step, which is why the last qualification is stated separately. □

Corollary 10.2 (Projected infrared equation)**.** *If all hypotheses of theorem 10.1 except tangent completeness hold, then the valid conclusion is*

$$(D\mathfrak{R}^{(g)}_{\mu,q})^* \left[G + \Lambda g + H^{\mathrm{loc}} + H^{\mathrm{nl}} - 8\pi G_{\mathrm{N}} T\right] = 0. \tag{75}$$

The theorem is conditional in two independent ways. It does not construct $\mathfrak{R}_\mu$, and it does not infer tangent completeness from the mere existence of $\mathfrak{R}_\mu$. It also assumes the massless spectrum rather than deriving it from information geometry. These are physical reconstruction questions, not identities of metric variation.

11 Derivative counting and coefficient assumptions

On a background whose characteristic curvature is

$$R_{\mu\nu\rho\sigma} \sim L^{-2}, \qquad \nabla \sim L^{-1}, \tag{76}$$

the Einstein tensor scales as L^{-2} and the tensors in equations (36) and (37) scale as L^{-4}. A six-derivative tensor scales as L^{-6}. It is convenient to factor the Wilson coefficients as

$$16\pi G_{\mathrm{N}} c_i = \widehat{c}_i \ell_*^2 \tag{77}$$

in four dimensions. Then

$$H^{\mathrm{loc}}_{\mu\nu} \sim L^{-2}\left[\widehat{c}_2 \frac{\ell_*^2}{L^2} + \widehat{c}_4 \frac{\ell_*^4}{L^4} + \cdots\right], \tag{78}$$

where the labels on $\widehat{c}$ indicate relative derivative order, not the operator index used earlier.

The statement that the Einstein tensor is leading requires more than $L \gg \ell_*$. It requires

$$|\widehat{c}_{2n}| \left(\frac{\ell_*}{L}\right)^{2n} \ll 1 \tag{79}$$

for the operators being neglected. A large number of species, proximity to a critical point, or a parametrically large bare coefficient may invalidate equation (79). In that case the correct leading equation includes the enhanced operator even at small curvature.

Field redefinitions introduce another qualification. At a fixed order in a local effective expansion, terms proportional to the lower-order equations of motion can be moved among operator coefficients. For example, a perturbative redefinition

$$g_{\mu\nu} \mapsto g_{\mu\nu} + a\ell_*^2 R_{\mu\nu} + b\ell_*^2 R g_{\mu\nu} \tag{80}$$

changes the coefficients of curvature-squared and matter-curvature operators. Therefore an individual c_i is generally scheme and basis dependent. The existence of a controlled expansion and on-shell observables are the invariant content. The explicit basis in equation (21) is used to audit signs and normalization, not to assert that each coefficient is separately measurable.

The decoupling of massive particles gives local operators at momenta below their masses under the usual assumptions [11]. Gravity itself remains an effective theory because the Einstein-Hilbert coupling has negative mass dimension in four dimensions. This does not prevent controlled low-energy predictions: nonrenormalizable interactions are organized order by order, as emphasized in the gravitational analysis of Donoghue [13] and in the general effective-Lagrangian viewpoint of Weinberg [10].

Proposition 11.1 (Constant-curvature estimate)**.** *Let $d > 2$ and let the background be maximally symmetric,*

$$R_{\mu\nu} = \frac{R}{d} g_{\mu\nu}, \qquad R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}). \tag{81}$$

Then the derivative terms in $H^{(1)}$ and $H^{(2)}$ vanish and

$$H^{(1)}_{\mu\nu} = \frac{4-d}{2d} R^2 g_{\mu\nu}, \tag{82}$$

$$H^{(2)}_{\mu\nu} = \frac{4-d}{2d^2} R^2 g_{\mu\nu}. \tag{83}$$

In $d = 4$ both tensors vanish identically on this background.

Proof. Insert equation (81) into equations (36) and (37). Since R is constant, all covariant derivatives of R and $R_{\mu\nu}$ vanish. The contraction

$$R_{\mu\alpha\nu\beta} R^{\alpha\beta} = \frac{R^2}{d^2} g_{\mu\nu} \tag{84}$$

then gives equations (82) and (83). □

12 Dimension and Lovelock qualifications

Lovelock's classification concerns symmetric divergence-free natural tensors built locally from a metric with second-order field equations [7, 8]. It is not a statement that every diffeomorphism-invariant action in every dimension reduces to the Einstein-Hilbert action. Higher-curvature actions generally have higher-order metric equations. Lovelock densities are special combinations whose field equations remain second order.

The kth Lovelock density can be written

$$\mathcal{L}_k = \frac{1}{2^k} \delta^{\mu_1\nu_1\cdots\mu_k\nu_k}_{\alpha_1\beta_1\cdots\alpha_k\beta_k} \prod_{j=1}^{k} R^{\alpha_j\beta_j}{}_{\mu_j\nu_j}. \tag{85}$$

The cases $k = 0$ and $k = 1$ are the cosmological and Einstein-Hilbert operators. The case $k = 2$ is $\mathcal{X}_4$ in equation (38).

The antisymmetric generalized Kronecker delta implies the following dimension dependence:

(a) $\mathcal{L}_k$ vanishes identically for $d < 2k$;

(b) its integral is topological, up to boundary completion, for $d = 2k$;

(c) it gives a nontrivial second-order metric tensor for $d > 2k$.

Thus the Gauss-Bonnet density has no local bulk metric equation in four dimensions but contributes in $d \geq 5$. The Einstein-Hilbert integral is topological in $d = 2$.

Proposition 12.1 (Four-dimensional local two-derivative form)**.** *In $d = 4$, a local symmetric divergence-free natural rank-two tensor depending on the metric and at most its first two derivatives, with the hypotheses of Lovelock's classification, is a linear combination of $G_{\mu\nu}$ and $g_{\mu\nu}$. If the coefficient of $G_{\mu\nu}$ is nonzero, normalization gives the left side of equation (74).*

The nonzero coefficient is a physical assumption. A pure cosmological action has no massless spin-2 kinetic term. A pure R^2 action about flat space does not supply the same one-graviton spectrum. The one-massless-spin-2 hypothesis and a nondegenerate two-derivative kinetic term rule out these cases.

For $d > 4$, higher Lovelock tensors can appear without producing derivatives above second order. Their actions nevertheless contain higher powers of curvature, and in an effective theory with controlled coefficients they are suppressed on a weakly curved background by powers of ℓ_*^2/L^2. Calling the field equations "second order" does not make those operators the same order in the curvature expansion. Conversely, if a Lovelock coefficient is parametrically large, its tensor must be retained.

In $d = 3$, the Riemann tensor is algebraically determined by the Ricci tensor, and pure Einstein gravity has no local propagating graviton. The Einstein equation is still a valid metric equation, but the phrase "one local massless spin-2 polarization" must be reformulated. Topologically massive gravity and other three-dimensional theories also show that additional derivative terms can change the spectrum. In $d = 2$, the Einstein tensor vanishes identically, so ordinary Einstein-Hilbert dynamics cannot be the leading propagating metric law. Dilaton gravity or other additional fields are needed for local dynamics. These cases are not counterexamples to theorem 10.1; they fall outside its spectral assumptions.

13 Massless loops and nonlocal corrections

Integrating out a field of mass m at momenta $p^2 \ll m^2$ normally gives a local expansion in p^2/m^2. A massless propagator has no such analytic threshold expansion. Its loop amplitudes contain nonanalytic terms such as

$$p^4 \log\left(\frac{-p^2 - i0}{\mu^2}\right), \qquad \sqrt{-p^2 - i0}, \tag{86}$$

depending on the process and dimension. In position space these correspond to nonlocal kernels. Their long-distance parts cannot be reproduced by adjusting a finite number of local Wilson coefficients. This separation is a central feature of gravitational effective field theory [13].

At quadratic order in curvature, a schematic covariant nonlocal action is

$$\begin{aligned}\Gamma_{\mathrm{nl}} = \int_M \mathrm{d}^4x\sqrt{-g}\Big[&b_1 R \log\left(\frac{-\Box - i0}{\mu^2}\right) R \\ &+ b_2 R_{\mu\nu} \log\left(\frac{-\Box - i0}{\mu^2}\right) R^{\mu\nu} \\ &+ b_3 R_{\mu\nu\rho\sigma} \log\left(\frac{-\Box - i0}{\mu^2}\right) R^{\mu\nu\rho\sigma} + \cdots\Big].\end{aligned} \tag{87}$$

The precise covariant completion, Green-function prescription, and coefficient basis depend on the quantum state and observable under consideration. The curvature expansion and its nonlocal form factors have been developed using covariant perturbation theory and related methods [12, 16, 15].

The notation $\log(-\Box/\mu^2)$ represents an integral kernel, not an ordinary pointwise function. In an in-out effective action the Feynman prescription is natural. A causal expectation-value equation generally requires the in-in, or Schwinger-Keldysh, effective action and retarded kernels. Varying an in-out functional and then interpreting the result as a causal equation can give the wrong boundary prescription even when its formal tensor structure is correct.

Define the nonlocal tensor by equation (71) rather than by an ellipsis. This has three advantages. It fixes its normalization relative to $8\pi G_{\mathrm{N}} T_{\mu\nu}$. It makes clear that metric variation acts both on the curvatures and on the form factor $\log(-\Box/\mu^2)$. It also preserves the Noether identity

$$\nabla^\mu H^{\mathrm{nl}}_{\mu\nu} = 0 \tag{88}$$

when the nonlocal functional and its boundary prescription are diffeomorphism invariant.

Massless-loop terms are often numerically small, but their suppression is not the same as the analytic decoupling of a heavy particle. Their coefficients can be calculable from the low-energy massless spectrum, and their nonanalyticity is invariant under local counterterm redefinitions. The correct infrared statement is therefore not that all corrections are local R^2 terms. It is that the Einstein operator is the leading local two-derivative term, accompanied by controlled local higher-derivative terms and by the nonlocal terms dictated by massless propagation.

14 Noether identity and stress-tensor compatibility

Let $\Gamma[g, \Phi]$ be invariant under a compactly supported infinitesimal diffeomorphism generated by ξ^μ. Its field variations are

$$\delta_\xi g^{\mu\nu} = -2\nabla^{(\mu}\xi^{\nu)}, \qquad \delta_\xi \Phi = \mathcal{L}_\xi \Phi. \tag{89}$$

Write the matter Euler-Lagrange expressions collectively as $\mathcal{E}_\Phi$. Diffeomorphism invariance gives

$$0 = \delta_\xi \Gamma = \int_M \mathrm{d}^d x \sqrt{-g} \left[\frac{1}{16\pi G_\mathrm{N}} \mathcal{C}_{\mu\nu}(-2\nabla^\mu \xi^\nu) + \mathcal{E}_\Phi \mathcal{L}_\xi \Phi \right], \tag{90}$$

where the definition of $\mathcal{C}$ includes the matter metric variation. Integration by parts yields a differential identity relating $\nabla^\mu \mathcal{C}_{\mu\nu}$ to the matter equations. On the matter shell,

$$\nabla^\mu \mathcal{C}_{\mu\nu} = 0. \tag{91}$$

Separating gravitational and matter functionals gives two familiar statements. Diffeomorphism invariance of the gravitational action implies

$$\nabla^\mu \mathcal{E}_{\mu\nu} = 0 \tag{92}$$

as an off-shell Noether identity. Diffeomorphism invariance of the matter action implies

$$\nabla^\mu T_{\mu\nu} = 0 \tag{93}$$

when the matter equations hold. For the Einstein-Hilbert term, equation (92) reduces to the contracted Bianchi identity. For the higher-curvature terms it gives equation (41); the general covariant phase-space analysis is developed by Iyer and Wald [14].

It is therefore imprecise to say that the Bianchi identity alone creates matter conservation. The geometric identity makes the metric equation compatible with conservation. The matter Ward identity, together with the matter equations and the absence of an anomaly, supplies equation (93). If the quantum matter theory has a diffeomorphism anomaly, its Ward identity contains an anomalous term. Then the effective description must cancel the anomaly by additional degrees of freedom or inflow, or else the hypotheses of theorem 10.1 fail.

Proposition 14.1 (Gauge directions do not test independent equations)**.** *Suppose equation (91) holds and ξ^μ has compact support or preserves the boundary data. Then*

$$\int_M \mathrm{d}^d x \sqrt{-g}\, \mathcal{C}_{\mu\nu} \mathcal{L}_\xi g^{\mu\nu} = 0. \tag{94}$$

Proof. Use $\mathcal{L}_\xi g^{\mu\nu} = -2\nabla^{(\mu}\xi^{\nu)}$, integrate by parts, and apply equation (91). The boundary term vanishes by hypothesis. □

This proposition explains why tangent completeness is imposed only modulo diffeomorphisms. Gauge variations lie in the null directions of the variational pairing once the Noether identity holds.

15 The one-massless-spin-2 assumption

The free massless spin-2 field $h_{\mu\nu}$ on Minkowski space is described by the Fierz-Pauli kinetic action and the gauge symmetry

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{95}$$

[2]. Consistent self-coupling to the conserved stress tensor requires the gravitational field's own stress to participate. Iterating this requirement, or imposing gauge invariance of the interacting theory, leads to the nonlinear structure of general relativity under the standard locality and field-content assumptions [3, 4].

For the present theorem, this argument has a limited but important role. It justifies the Einstein-Hilbert kinetic term once the following are assumed:

(a) there is exactly one relevant massless helicity-two representation;

(b) it couples universally to a conserved stress tensor;

(c) the interactions are local in the effective-field-theory sense;

(d) the gauge symmetry deforms consistently into a single metric diffeomorphism redundancy;

(e) the two-derivative kinetic coefficient is nonzero and has the sign required for a healthy low-energy graviton.

It does not derive these assumptions from the microscopic state space.

The word "one" matters. With several massless spin-2 fields, cross-coupling consistency is restrictive, but the field space and possible decoupled sectors must be analyzed rather than replaced by a single metric by definition. With a massive spin-2 field, a mass term is a relevant infrared datum and the gauge structure differs. With a massless scalar, scalar-tensor interactions can contribute at the same long-distance order as Einstein exchange. With an independent connection or tetrad, torsion and nonmetricity may survive. Each case violates or modifies theorem 4.1.

The coefficient sign also matters. Write the quadratic action for a physical transverse-traceless perturbation schematically as

$$\Gamma^{(2)} \sim \frac{1}{64\pi G_{\mathrm{N}}} \int \mathrm{d}^d x\, h^{\mathrm{TT}} \Box h^{\mathrm{TT}} + cdots. \tag{96}$$

The precise sign depends on signature and integration-by-parts conventions, but positivity of physical residues selects the healthy sign of G_{N}. Lovelock classification alone gives an arbitrary coefficient multiplying $G_{\mu\nu}$; it does not establish that the coefficient is nonzero or positive.

16 Failure modes and counterexamples

The hypotheses of theorem 10.1 are easiest to understand by removing them one at a time.

16.1 No Lorentzian reconstruction

A generic finite spin system or topological phase need not admit any smooth Lorentzian metric description. The effective-action derivation then has no metric field on which to operate. Information geometry on a parameter manifold is not by itself a spacetime reconstruction: its signature, dimension, locality, and coordinate redundancy can all differ.

16.2 Restricted reconstruction tangent

The conformal example theorem 9.3 yields only the trace equation. A finite set of global moduli yields only integrated equations. A homogeneous ansatz yields only homogeneous components. These are not failures of the variational principle. They are correct projected equations for an insufficiently large test space.

16.3 Additional long-range fields

Consider a scalar-tensor action

$$\Gamma[g,\varphi] = \int \mathrm{d}^d x \sqrt{-g}\left[\frac{F(\varphi)}{2} R - \frac{1}{2}(\nabla\varphi)^2 - V(\varphi)\right] + \Gamma_{\mathrm{m}}. \tag{97}$$

If φ is light, its equation and its contribution to the metric equation are not higher-derivative corrections. They survive in the infrared. The Einstein equation with a constant Newton coupling is not the complete leading law unless φ is stabilized or otherwise decouples.

16.4 Uncontrolled Wilson coefficients

For an action

$$\Gamma = \frac{1}{16\pi G_{\mathrm{N}}} \int \sqrt{-g}\, R + c_1 \int \sqrt{-g}\, R^2, \tag{98}$$

the ratio of the two metric tensors on scale L is

$$16\pi G_{\mathrm{N}} |c_1| L^{-2}. \tag{99}$$

No conclusion follows from L being macroscopically large unless this dimensionless product is small. A coefficient of order $L^2/(16\pi G_{\mathrm{N}})$ makes the R^2 equation competitive.

16.5 Massless nonlocality

Gapless matter may generate equation (87). The resulting kernel can be long ranged even though it begins at curvature-squared order. Replacing it by a local constant coefficient loses the nonanalytic information. The correct equation is equation (73), not its purely local truncation.

16.6 Dimension outside the four-dimensional classification

In $d \geq 5$, Gauss-Bonnet and higher Lovelock tensors may contribute. In $d = 2$, the Einstein tensor vanishes. In $d = 3$, the absence of local pure- Einstein graviton degrees of freedom changes the spectral premise. A theorem that omits dimension is therefore ambiguous.

16.7 Anomalous Ward identity

If the quantum effective action is not invariant under diffeomorphisms, then

$$\nabla^{\mu} T_{\mu\nu} = \mathcal{A}_{\nu} \tag{100}$$

for an anomaly $\mathcal{A}_{\nu}$. The metric Euler-Lagrange tensor of an invariant gravitational action remains divergence free, so the uncancelled equation is inconsistent. Anomaly cancellation or inflow is part of the admissibility conditions, not a consequence of stationarity.

17 Relation to reconstructed information geometry

The reconstruction map in equation (16) may begin with a smooth family of faithful quantum states $\rho(\lambda)$ and a specified monotone information metric $\mathcal{G}$. For example, the Hessian of Umegaki relative entropy with displacement convention $\mathrm{d}\lambda = \lambda' - \lambda$ is

$$\mathcal{G}^{\mathrm{BKM}}_{ab}(\lambda) = -\left.\frac{\partial^2}{\partial \lambda^a \partial \lambda'^b} D\big(\rho(\lambda)\|\rho(\lambda')\big)\right|_{\lambda'=\lambda}. \tag{101}$$

This is the Bogoliubov-Kubo-Mori metric. It should not be identified without qualification with the Bures metric or the symmetric-logarithmic-derivative Fisher metric. The distinction matters especially at changes of rank.

Nothing in the metric variation derived above depends on choosing equation (101). What matters is that the selected microscopic data, together with locality and coarse-graining structure, supply a smooth map to $g_{\mu\nu}$. The information metric can be part of the construction of that map, but the Einstein-Hilbert variation takes place in the reconstructed field space.

This distinction also locates the present result relative to thermodynamic and entanglement-based approaches to gravitational dynamics. Jacobson derived the Einstein equation as a local horizon equation of state [17]. In holographic conformal field theories, the entanglement first law for boundary balls can be equivalent to the linearized bulk gravitational equations [18, 19]. Entanglement structure has also been used to reconstruct spatial geometry and a spatial analogue of the Einstein equation [20]. Each construction supplies particular microscopic variables and particular admissible perturbations. The tangent completeness condition here identifies the additional functional requirement needed before stationarity in such variables can imply every local component of a spacetime metric equation.

The most direct connection between microscopic distinguishability and the metric equation is therefore the tangent map

$$D\mathfrak{R}^{(g)}_{\mu,q} : \delta q \longmapsto \delta g. \tag{102}$$

It determines which metric components microscopic state variations can test. If the image is complete modulo gauge, reconstruction stationarity implies the full equation. If not, it implies a projection. This formulation avoids the unsupported step of treating every variation of an information metric as an arbitrary local variation of a Lorentzian spacetime metric.

The reconstruction scale μ also should not be confused with a metric equation. Wilson coefficients run with μ, and field redefinitions can move scale dependence between couplings and fields. A separate beta functional for the reconstructed metric requires a local renormalization-group structure and a specified relation between that beta functional and the effective action. No such relation is assumed in theorem 10.1.

18 Scope of the result

The infrared consistency equation has a conventional status. It is the Euler-Lagrange equation of a diffeomorphism-invariant effective action, pulled back through a reconstruction map. Its Einstein form at leading local two-derivative order depends on the field content, coefficient hierarchy, dimension, and completeness of admissible variations.

The following statements are not consequences of the analysis:

(1) An arbitrary density matrix determines a spacetime metric.

(2) Every quantum system has a Lorentzian continuum limit.

(3) Information-metric positivity alone implies the nonlinear Einstein equation.

(4) A finite reconstruction ansatz tests all local metric equations.

(5) Lovelock's theorem excludes every higher-curvature correction.

(6) Massive-field decoupling removes nonlocal effects of massless fields.

(7) A symbolic or numerical implementation proves the continuum variational identities.

What does follow is narrower and useful. Once a reconstruction is known to produce the correct variational field space, the low-energy metric equation is not an additional guess. It is determined by stationarity of the effective action. Under a one-metric massless-spin-2 spectrum and controlled power counting, its leading local kinetic term is Einstein-Hilbert. The remaining terms have a definite place: the cosmological term remains leading, higher-curvature operators enter with their Wilson coefficients, Lovelock terms depend on dimension, and massless loops supply nonlocal form factors.

19 Conclusion

For a reconstructed metric, the correct stationarity statement is initially an equation on reconstruction space. Its tensor form is $(D\mathfrak{R})^*\mathcal{C} = 0$. Promoting it to $\mathcal{C}_{\mu\nu} = 0$ requires the explicit hypothesis that admissible reconstruction variations span physical local metric variations. This condition is the missing functional step in many informal arguments from microscopic stationarity to a gravitational field equation.

With that step supplied, the metric variation is standard and coefficient complete. The Einstein-Hilbert and cosmological terms give $G_{\mu\nu}+\Lambda g_{\mu\nu}$, the matter convention gives $8\pi G_{\mathrm{N}} T_{\mu\nu}$ on the right side, and curvature-squared and nonlocal functionals give tensors normalized by $16\pi G_{\mathrm{N}}$ times their functional derivatives. The Gibbons-Hawking-York term resolves the smooth non-null Dirichlet boundary problem for the Einstein-Hilbert sector, while other boundary types and higher-curvature actions require their own completions.

The resulting infrared equation is Einstein's equation only inside a stated class: one interacting massless metric spin-2 mode, no other unsuppressed long-range fields, a controlled derivative expansion, appropriate dimension, anomaly-free diffeomorphism redundancy, and a reconstruction tangent large enough to test the equation. Outside that class, the projected equation or an enlarged infrared field theory is the appropriate conclusion.

A Variation identities

For reference, the variation of the Levi-Civita connection with respect to the covariant metric is

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\nabla_\mu \delta g_{\sigma\nu} + \nabla_\nu \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\mu\nu}\right). \tag{103}$$

Together with equation (24), this gives

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}. \tag{104}$$

After two integrations by parts,

$$\begin{aligned}\delta\int\sqrt{-g}\,R^2 = \int\sqrt{-g}\big[&2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2\\ &+ 2g_{\mu\nu}\Box R - 2\nabla_\mu\nabla_\nu R\big]\delta g^{\mu\nu},\end{aligned} \tag{105}$$

which is equation (36).

For the Ricci-squared operator, begin with

$$\delta(R_{\alpha\beta}R^{\alpha\beta}) = 2R_\mu{}^\alpha R_{\nu\alpha}\delta g^{\mu\nu} + 2R^{\mu\nu}\delta R_{\mu\nu}. \tag{106}$$

Using equation (24), integrating by parts, and commuting covariant derivatives produces curvature commutators. The algebraic $R_\mu{}^\alpha R_{\nu\alpha}$ pieces combine with those commutators, leaving equation (37). An equivalent form can be obtained using the contracted Bianchi identity. Equality of such forms depends on using a single Riemann-sign convention throughout.

The traces in general dimension are

$$g^{\mu\nu}H^{(1)}_{\mu\nu} = \frac{4-d}{2}R^2 + 2(d-1)\Box R, \tag{107}$$

$$g^{\mu\nu}H^{(2)}_{\mu\nu} = \frac{4-d}{2}R_{\alpha\beta}R^{\alpha\beta} + \frac{d}{2}\Box R. \tag{108}$$

For $d = 4$, these reduce to the checks quoted in section 6.

B Gauss-Bonnet check in four dimensions

For a compact oriented four-manifold without boundary, the Chern-Gauss-Bonnet theorem gives

$$\frac{1}{32\pi^2}\int_M \mathrm{d}^4x\sqrt{|g|}\,\mathcal{X}_4 = \chi(M), \tag{109}$$

with the corresponding signature adjustment understood for a Lorentzian continuation. Since the Euler characteristic is unchanged by a smooth metric variation, the bulk functional derivative vanishes:

$$H^{(\mathrm{GB})}_{\mu\nu} = 0 \qquad (d = 4). \tag{110}$$

Combining equation (110) with equation (40) gives

$$H^{(3)}_{\mu\nu} = 4H^{(2)}_{\mu\nu} - H^{(1)}_{\mu\nu} \qquad (d = 4). \tag{111}$$

This identity removes one curvature-squared bulk operator from a four-dimensional local basis. With a boundary, the topological invariant also contains its boundary completion, so discarding the bulk density without tracking the boundary term is not legitimate for boundary observables.

C Symbolic power-counting model

The companion Haskell modules implement only the coefficient arithmetic and finite-dimensional projection examples used for diagnostics. A local term is represented by its relative number of derivatives, a dimensionless Wilson coefficient, and $\epsilon = \ell_*/L$. Its relative size is modeled as

$$r_n = |\widehat{c}_n|\epsilon^n. \tag{112}$$

For curvature-squared corrections relative to Einstein-Hilbert, $n = 2$. The implementation also represents a finite defect vector C and a matrix R whose columns are admissible reconstruction variations. Stationarity is the finite-dimensional equation

$$R^{\mathsf{T}} C = 0. \tag{113}$$

A full-rank R forces $C = 0$, while a rank-deficient R can annihilate a nonzero defect. This is an executable analogy for theorems 9.1 and 9.2; it is not a proof of the continuum theorem or of any microscopic reconstruction hypothesis.

References

[1] E. Noether, "Invariante Variationsprobleme," Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, pp. 235–257 (1918). English translation by M. A. Tavel, Transport Theory and Statistical Physics **1**, 186–207 (1971), doi:10.1080/00411457108231446.

[2] M. Fierz and W. Pauli, "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field," Proceedings of the Royal Society A **173**, 211–232 (1939), doi:10.1098/rspa.1939.0140.

[3] S. Weinberg, "Photons and gravitons in S-matrix theory: derivation of charge conservation and equality of gravitational and inertial mass," Physical Review **135**, B1049–B1056 (1964), doi:10.1103/PhysRev.135.B1049.

[4] S. Deser, "Self-interaction and gauge invariance," General Relativity and Gravitation **1**, 9–18 (1970), doi:10.1007/BF00759198, arXiv:gr-qc/0411023.

[5] J. W. York, Jr., "Role of conformal three-geometry in the dynamics of gravitation," Physical Review Letters **28**, 1082–1085 (1972), doi:10.1103/PhysRevLett.28.1082.

[6] G. W. Gibbons and S. W. Hawking, "Action integrals and partition functions in quantum gravity," Physical Review D **15**, 2752–2756 (1977), doi:10.1103/PhysRevD.15.2752.

[7] D. Lovelock, "The Einstein tensor and its generalizations," Journal of Mathematical Physics **12**, 498–501 (1971), doi:10.1063/1.1665613.

[8] A. Navarro and J. Navarro, "Lovelock's theorem revisited," Journal of Geometry and Physics **61**, 1950–1956 (2011), doi:10.1016/j.geomphys.2011.05.004, arXiv:1005.2386.

[9] K. S. Stelle, "Renormalization of higher-derivative quantum gravity," Physical Review D **16**, 953–969 (1977), doi:10.1103/PhysRevD.16.953.

[10] S. Weinberg, "Phenomenological Lagrangians," Physica A **96**, 327–340 (1979), doi:10.1016/0378-4371(79)90223-1.

[11] T. Appelquist and J. Carazzone, "Infrared singularities and massive fields," Physical Review D **11**, 2856–2861 (1975), doi:10.1103/PhysRevD.11.2856.

[12] A. O. Barvinsky and G. A. Vilkovisky, "Beyond the Schwinger-DeWitt technique: converting loops into trees and in-in currents," Nuclear Physics B **282**, 163–188 (1987), doi:10.1016/0550-3213(87)90681-X.

[13] J. F. Donoghue, "General relativity as an effective field theory: the leading quantum corrections," Physical Review D **50**, 3874–3888 (1994), doi:10.1103/PhysRevD.50.3874, arXiv:gr-qc/9405057.

[14] V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy," Physical Review D **50**, 846–864 (1994), doi:10.1103/PhysRevD.50.846, arXiv:gr-qc/9403028.

[15] J. F. Donoghue and B. K. El-Menoufi, "Covariant non-local action for massless QED and the curvature expansion," arXiv:1507.06321 [hep-th] (2015).

[16] A. Codello and R. K. Jain, "On the covariant formalism of the effective field theory of gravity and leading order corrections," Classical and Quantum Gravity **33**, 225006 (2016), doi:10.1088/0264-9381/33/22/225006, arXiv:1507.06308.

[17] T. Jacobson, "Thermodynamics of spacetime: the Einstein equation of state," Physical Review Letters **75**, 1260–1263 (1995), doi:10.1103/PhysRevLett.75.1260, arXiv:gr-qc/9504004.

[18] N. Lashkari, M. B. McDermott, and M. Van Raamsdonk, "Gravitational dynamics from entanglement 'thermodynamics'," Journal of High Energy Physics **2014**, 195 (2014), doi:10.1007/JHEP04(2014)195, arXiv:1308.3716.

[19] T. Faulkner et al., "Gravitation from entanglement in holographic CFTs," Journal of High Energy Physics **2014**, 51 (2014), doi:10.1007/JHEP03(2014)051, arXiv:1312.7856.

[20] C. Cao, S. M. Carroll, and S. Michalakis, "Space from Hilbert space: recovering geometry from bulk entanglement," Physical Review D **95**, 024031 (2017), doi:10.1103/PhysRevD.95.024031, arXiv:1606.08444.