Infrared Axioms for Reconstructed Lorentzian Geometry

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Abstract

Suppose that coarse-grained quantum data admit a reconstruction as a smooth Lorentzian metric and additional infrared fields. This supposition does not by itself determine the dynamics or the gauge structure of the reconstructed variables. We formulate an axiom system that separates the assumptions needed in later infrared arguments. The axioms concern reconstruction regularity, infrared locality, Lorentzian kinematics, relabeling redundancy, variational completeness, stress-tensor conservation, absence of diffeomorphism anomalies, the entanglement first law, relative-entropy positivity, separation of scales, the long-range spectrum, and dimension-dependent classification. The relabeling axiom contains a faithful action of spacetime diffeomorphisms on representatives of the same reconstructed observables. It is additional structure and is not implied by a monotone information metric. We prove a dependency theorem for the two-derivative effective action and give remove-one-axiom countermodels. The countermodels include nonlocal metric actions, preferred-coordinate reconstructions, anomalous Ward identities, scalar-tensor spectra, topological infrared limits, and cases with uncontrolled Wilson coefficients. Entanglement first-law identities and relative-entropy positivity are retained as separate informational hypotheses because they constrain different orders in a state perturbation and neither supplies the effective gravitational dictionary. The resulting framework is conditional: it describes the class in which a local diffeomorphism-redundant metric can be treated as an infrared field, without asserting that a generic quantum system belongs to that class.

Contents

1 Introduction

Information geometry associates bilinear forms with smooth families of probability distributions or quantum states. In quantum theory there is a family of monotone metrics, rather than a unique quantum Fisher metric, as emphasized by Petz and Sudar [1, 2]. Such a metric measures distinguishability along a state family. Its indices label state-space parameters. They are not spacetime tensor indices unless a further reconstruction dictionary identifies suitable collective directions with geometric perturbations.

This distinction becomes decisive when the output of a coarse-graining procedure is written as a Lorentzian metric. A formula

$$\mathfrak{R}_\mu : (\mathcal{A}, \rho, \mathcal{G}) \longmapsto (M, g, \Phi; \mathbf{c}(\mu)) \tag{1}$$

may summarize the intended output, but it leaves several logically different questions unanswered. Does a local algebra in the microscopic description induce local response in the reconstructed variables? Do different coordinate descriptions represent the same observables? Are all local metric variations represented by variations of microscopic states? Does a diffeomorphism Ward identity hold, and is it anomalous? Is the low-energy spectrum that of one massless spin-two mode, or are scalar, vector, and further tensor modes also present? Are higher-derivative terms actually small at the scale of interest?

The answers cannot be bundled into the word "emergent." They must be hypotheses with distinct uses. This paper supplies such a separation. Its purpose is preparatory. It does not derive a metric reconstruction from a generic quantum system and it does not yet vary an effective action to obtain a field equation. Instead it identifies a domain in which those later steps are meaningful.

The separation is also needed to state accurately what follows from entanglement. For a differentiable family of faithful reduced states, the first variation of relative entropy gives

$$\delta S_B = \delta \langle K_B \rangle. \tag{2}$$

This is a kinematic identity in state space [4]. In holographic settings it contributes to a derivation of linearized bulk constraints only after entropy-area and stress-tensor dictionaries are supplied [5, 6]. In the small-ball argument it is combined with fixed-volume entanglement stationarity, a locally vacuum-like modular Hamiltonian, and assumptions about matter [7]. Positivity of relative entropy is a separate second-order statement that, in a holographic code subspace, becomes positivity of bulk canonical energy [9]. Neither statement creates diffeomorphism redundancy or selects the infrared spectrum.

The organization is as follows. Section 2 fixes the reconstruction data and conventions. Section 3 states the axioms one at a time. Section 4 analyzes the distinction between state-space geometry and diffeomorphism redundancy. Section 5 derives the conditional form of a local derivative expansion. Section 6 separates the informational hypotheses. Section 7 treats conservation and anomaly freedom. Section 8 records dimensional and spectral caveats. Section 9 gives the main dependency theorem. Section 10 constructs remove-one-axiom countermodels. The appendices collect a formal dependency table, dimensional estimates, and a finite validation model.

2 Reconstruction data and conventions

2.1 Microscopic input

Let $\mathfrak{U}$ be a directed family of microscopic regions. The microscopic system carries an isotonic net of unital algebras

$$U \subseteq V \quad \Longrightarrow \quad \mathcal{A}(U) \subseteq \mathcal{A}(V). \tag{3}$$

When the system is relativistic, one may also impose commutativity for spacelike separated regions, following the algebraic framework of Haag and Kastler [3]. We do not assume at the outset that the microscopic regions already form a Lorentzian manifold. The symbol U can instead denote a lattice region, a subsystem selected by a tensor factorization, or an element of another localization structure.

Let $\lambda \mapsto \rho(\lambda)$ be a C^2 family of faithful states in a finite-dimensional chart of a state manifold $\mathcal{S}$. For definiteness, the information metric in this paper is the Bogoliubov-Kubo-Mori metric obtained from the mixed Hessian of Umegaki relative entropy:

$$\mathcal{G}_{ab}(\lambda) = -\left.\frac{\partial^2}{\partial\lambda^a \partial\lambda'^b} D\big(\rho(\lambda)\|\rho(\lambda')\big)\right|_{\lambda'=\lambda}. \tag{4}$$

The minus sign follows the convention that the derivatives act on the first and second arguments independently before restriction to the diagonal. Diagonal stationarity makes the mixed Hessian negative semidefinite relative to the second-argument Hessian, so the displayed minus sign gives a positive semidefinite BKM form, positive definite after any null directions are quotiented. Other Petz metrics may be used, but changing the metric changes the input data. No argument below identifies the Bures, symmetric logarithmic derivative, and Bogoliubov-Kubo-Mori metrics.

2.2 Reconstructed output

At a coarse-graining scale μ, a reconstruction assigns

$$\mathfrak{R}_\mu(\mathcal{A}, \rho, \mathcal{G}) = \big[M, g, \Phi; \mathbf{c}(\mu)\big]. \tag{5}$$

Here M is a smooth connected d-manifold, g is a smooth metric of signature $(-, +, \ldots, +)$, Φ denotes all other retained fields, and $\mathbf{c}(\mu)$ denotes Wilsonian couplings. Square brackets anticipate an equivalence relation on representatives. The equivalence relation will be specified by an independent redundancy axiom.

For a reconstructed region $O \subset M$, write $\mathcal{A}_\mu(O)$ for the microscopic subalgebra assigned to it by the reconstruction. For a state perturbation J supported in a region U of the microscopic localization structure, let

$$\delta_J g_{\mu\nu}(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t} g_{\mu\nu}[\rho + tJ](x)\right|_{t=0} \tag{6}$$

whenever the derivative exists. J is a trace-zero density-operator variation in the tangent space of the state family, not an external classical source coupled to an infrared field. This notation is schematic in infinite dimensions, but it makes the support question precise: locality concerns the response kernel of the reconstruction, not merely commutativity of the microscopic algebras.

Definition 2.1 (Reconstruction record)**.** A reconstruction record at scale μ is a tuple

$$\mathfrak{X}_\mu = (\mathcal{A}, \mathcal{S}, \mathcal{G}, \mathfrak{R}_\mu, M, g, \Phi, \mathbf{c}; \epsilon, \ell_*, L), \tag{7}$$

where $\epsilon = \ell_*/L$ is the proposed derivative-expansion parameter. The record contains data only. It is not called admissible until specified axioms have been checked.

Definition 2.2 (Implication between axioms)**.** For axioms A and B, the notation $A \Rightarrow B$ means that every reconstruction record satisfying A also satisfies B, without adding a dictionary or a dynamical premise. A countermodel to $A \Rightarrow B$ is a record satisfying A and violating B.

This semantic convention prevents an argument from hiding extra input in a change of terminology. In particular, calling state parameters "coordinates" does not establish a spacetime action of $\mathrm{Diff}(M)$.

3 The infrared axiom system

The labels in this section are local to this paper. They are chosen to make dependency statements compact. Each axiom has a different mathematical type. Some are properties of the reconstruction map, some of an effective action, some of the state family, and some of the spectrum.

3.1 Regularity and kinematics

Axiom 3.1 (R: regular reconstruction)**.** There is a scale interval I_μ on which (1) assigns a smooth d-manifold, a nondegenerate Lorentzian metric, and smooth infrared fields. On the chosen constant-rank stratum of $\mathcal{S}$, the map from state perturbations to field perturbations is at least twice Fréchet differentiable in the norms used for the infrared expansion.

Regularity contains both existence and smoothness. It does not contain locality, gauge redundancy, or a claim that every state-space direction is geometric. The constant-rank restriction avoids using a Riemannian expansion across a singular rank-changing boundary.

Axiom 3.2 (L: infrared locality)**.** There is a reconstruction localization map $U \mapsto O_\mu(U) \subset M$ and a length ℓ_* such that a microscopic perturbation supported in U changes a reconstructed observable at x by a kernel that decays when $\mathrm{dist}(x, O_\mu(U)) \gg \ell_*$. After fields above the gap are integrated out, the analytic part of the effective action is an integral of local scalar densities, ordered by derivatives. Nonanalytic contributions from retained massless fields are recorded separately.

The last sentence is essential. Massless loops generate nonlocal and nonanalytic terms even in a valid gravitational effective field theory [10]. Thus locality here does not say that the exact quantum effective action is a finite polynomial in derivatives. It says that a local Wilsonian part exists and that nonlocal terms have a controlled origin.

Axiom 3.3 (D: reconstruction redundancy)**.** There is an equivalence relation $\sim$ on reconstructed representatives and a faithful action

$$\mathrm{Diff}(M) \times \mathrm{Rep}(\mathfrak{X}_\mu) \longrightarrow \mathrm{Rep}(\mathfrak{X}_\mu) \tag{8}$$

such that

$$(M, g, \Phi) \sim (M, \varphi^* g, \varphi^* \Phi) \qquad \text{for every } \varphi \in \mathrm{Diff}(M), \tag{9}$$

and all reconstructed observables factor through the quotient by this action. The action is compatible with composition and with the localization map.

The faithfulness condition concerns the action on representatives, not on physical equivalence classes, where all gauge transformations act trivially by definition. It rules out an arbitrary parameter relabeling that fails to act as a pullback on the reconstructed tensor fields.

Axiom 3.4 (V: variational completeness)**.** At each admissible background, the image of the differential of $\mathfrak{R}_\mu$ contains the compactly supported local metric variations used to test the infrared action, modulo infinitesimal diffeomorphisms and any stated boundary conditions. Equivalently, if a covariant distribution $E^{\mu\nu}$ obeys

$$\int_M \sqrt{-g}\, E^{\mu\nu} \delta g_{\mu\nu} = 0 \tag{10}$$

for every reconstruction-induced admissible variation, then $E^{\mu\nu} = 0$ as a functional on gauge-inequivalent local metric variations.

Variational completeness is stronger than differentiability. A reconstruction may produce a smooth one-parameter family of metrics while sampling only one global conformal mode. Stationarity along that family would give one integrated equation, not the local metric Euler-Lagrange equations.

3.2 Ward and anomaly hypotheses

Axiom 3.5 (C: on-shell matter conservation)**.** For the retained matter fields on their effective equations of motion, the chosen stress tensor satisfies

$$\nabla^\mu T_{\mu\nu} = 0. \tag{11}$$

The stress tensor is defined by metric variation with the convention

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \Gamma_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{12}$$

Here and below, $T_{\mu\nu}$ denotes the renormalized effective expectation value obtained from the matter effective action.

This axiom states the consequence needed by the later field equation. It does not identify the reason it holds. The reason is supplied by the next axiom together with the matter equations of motion.

Axiom 3.6 (A: anomaly-free diffeomorphism Ward identity)**.** The renormalized infrared generating functional and its measure admit a diffeomorphism-invariant scheme on the field content and backgrounds under consideration. For compactly supported vector fields ξ, the quantum Ward identity has no local anomaly term:

$$\delta_\xi \Gamma_{\mathrm{IR}} = 0. \tag{13}$$

If a boundary is present, boundary inflow and boundary degrees of freedom are included so that the combined system obeys the corresponding identity.

Anomaly freedom is not synonymous with classical covariance. Chiral theories in suitable dimensions can have gravitational anomalies [11]. An anomalous theory may still be written with classically covariant fields, but its quantum effective action does not satisfy (13).

3.3 Information-theoretic hypotheses

Axiom 3.7 (F: differentiable entanglement first law)**.** For each region B in a specified class and a faithful reference state σ_B, the reduced-state family remains in a differentiable domain of $K_B = -\log \sigma_B$ and satisfies

$$\delta S_B = \delta \langle K_B \rangle \tag{14}$$

for all allowed first-order state perturbations of unit trace. Any local formula for K_B is additional data and is not part of this axiom.

Axiom 3.8 (P: relative-entropy positivity and monotonicity)**.** For the same algebraic inclusions and faithful states,

$$D(\rho_B \| \sigma_B) \geq 0, \tag{15}$$

$$B_1 \subseteq B_2 \quad \Longrightarrow \quad D(\rho_{B_1} \| \sigma_{B_1}) \leq D(\rho_{B_2} \| \sigma_{B_2}). \tag{16}$$

The quadratic Hessian is taken only in directions where the expansion exists.

The first law and positivity are stated separately even though both follow from standard relative entropy in their common finite-dimensional domain. They enter gravitational arguments at different orders and require different additional dictionaries. The logical atom used in a proof should be the exact property invoked, not a larger package imported without comment.

3.4 Scale and spectrum hypotheses

Axiom 3.9 (S: scale separation)**.** There are lengths ℓ_* and L such that

$$\epsilon = \frac{\ell_*}{L} \ll 1, \tag{17}$$

and modes integrated out of the Wilsonian action have masses of order ℓ_*^{-1} or larger. The observables and backgrounds considered vary on scales of order L. Thresholds of retained gapless fields are not counted as heavy.

Axiom 3.10 (N: controlled Wilson coefficients)**.** In units set by ℓ_*, the Wilson coefficients of higher-derivative local operators are bounded so that the nominal derivative counting is not offset by large coefficients on the backgrounds considered. For example,

$$|\alpha_i|\, R \ll \frac{1}{G} \tag{18}$$

in the normalization used for a curvature-squared term. The cosmological term is not declared small by derivative counting.

Scale separation alone is insufficient. A coefficient proportional to ϵ^{-2} can make a four-derivative operator as important as the two-derivative term. Conversely, a small coefficient can extend the validity of the expansion beyond a naive estimate.

Axiom 3.11 (Q: infrared spectrum)**.** The long-range geometric sector contains one nondegenerate massless spin-two mode represented by $g_{\mu\nu}$. There are no additional unsuppressed massless spin-two fields, long-range torsion or nonmetricity modes, or scalar and vector fields that mix with the metric at the same derivative order unless they are explicitly included in Φ and retained in the conclusion. The sign of the physical spin-two kinetic term is positive.

This axiom excludes scalar-tensor and vector-tensor theories only when the desired conclusion is a pure metric leading law. Those theories are consistent effective theories in their own domains. Their existence is a reason to expose the spectrum assumption, not a defect in the effective field theory method.

Axiom 3.12 (M: dimension and topology domain)**.** The dimension d, topology, boundary conditions, and parity properties are fixed before applying any tensor-classification statement. When an Einstein kinetic term is required, assume $d \geq 3$. When local propagating metric polarizations are required, assume $d \geq 4$ and an appropriate background. Possible Lovelock densities, parity-odd terms, and boundary terms are included or excluded explicitly for that dimension.

4 Diffeomorphism redundancy is additional structure

The state manifold $\mathcal{S}$ and the reconstructed manifold M are different spaces. A coordinate change on $\mathcal{S}$ acts on the labels λ^a. A spacetime diffeomorphism acts on points of M and pulls back tensor fields. Even when $\dim \mathcal{S} = d$, equality of dimensions does not identify these actions.

Lemma 4.1 (State-chart covariance)**.** *Let $\mathcal{G}$ be a Riemannian metric on $\mathcal{S}$. Under a change of state coordinates $\lambda \mapsto \tilde{\lambda}(\lambda)$,*

$$\widetilde{\mathcal{G}}_{ij} = \frac{\partial \lambda^a}{\partial \tilde{\lambda}^i} \frac{\partial \lambda^b}{\partial \tilde{\lambda}^j} \mathcal{G}_{ab}. \tag{19}$$

This covariance gives an action of state-manifold chart changes on $\mathcal{G}$. It does not define an action of $\mathrm{Diff}(M)$ *on g.*

Proof. Equation (19) is the tensor transformation law on $\mathcal{S}$. To obtain a pullback action on M, one would need a map from each spacetime diffeomorphism to an automorphism of the reconstruction data that preserves all reconstructed observables. No such map occurs in the premises. Therefore the first action does not furnish the second. □

Proposition 4.2 (Necessary bridge for reconstructed gauge redundancy)**.** *Suppose the reconstruction assigns local observables $\mathcal{O}_i[\rho]$ to covariant functionals $O_i[g,\Phi]$. Reconstruction-coordinate independence yields diffeomorphism redundancy only if there is a group homomorphism*

$$\iota : \mathrm{Diff}(M) \longrightarrow \mathrm{Aut}(\mathcal{A}, \mathcal{S}, \mathfrak{R}_\mu) \tag{20}$$

such that

$$O_i[\varphi^* g, \varphi^* \Phi] = O_i[g,\Phi] \circ \iota(\varphi) \tag{21}$$

for every reconstructed observable in the defining family. If the family separates physical configurations, this intertwiner induces the equivalence (9).

Proof. The homomorphism supplies composition and the identity element. The intertwining equation says that changing the reconstructed representative can be compensated by acting on the microscopic description without changing physical predictions. Separation of physical configurations then makes the quotient well-defined. Without ι, a pullback of g is merely a second tensor field in the codomain of the reconstruction and need not describe the same microscopic state. □

Example 4.3 (Information geometry with a preferred chart)**.** Let $\mathcal{S}$ be the open probability simplex with the classical Fisher metric. Choose a fixed embedding $X : \mathcal{S} \to \mathbb{R}^d$ and define

$$g_{\mu\nu}(X(\lambda)) = \eta_{\mu\nu} + u_\mu u_\nu f(\lambda) \tag{22}$$

for a fixed covector u_μ and scalar f. The Fisher metric remains invariant under reparameterization of λ as a state-space tensor. The reconstruction nevertheless selects X and u as preferred background data. A generic pullback of g is not equivalent to the original output. Thus a well-defined monotone information metric coexists with failure of Theorem 3.3.

Remark 4.4. Holographic quantum error correction gives a concrete setting in which multiple boundary representations can encode one bulk operator [16, 17]. That example motivates a redundancy dictionary, but it does not prove that an arbitrary monotone state metric has one.

5 Local effective actions and derivative counting

Assume Theorems 3.1 to 3.3, 3.9 and 3.10. The local Wilsonian action can be organized by diffeomorphism-invariant scalar operators. With the stress convention (12), write

$$\Gamma_{\mathrm{IR}}[g,\Phi] = \Gamma_{\mathrm{grav}}[g,\Phi] + \Gamma_{\mathrm{matter}}[g,\Phi] + \Gamma_{\mathrm{nonlocal}}[g,\Phi]. \tag{23}$$

The split between gravitational and matter terms can be scheme dependent when fields mix. Only the full effective action is invariant under local field redefinitions.

For parity-even pure metric terms, the first orders have the form

$$\Gamma^{\mathrm{local}}_{\mathrm{grav}} = \int_M \mathrm{d}^d x \sqrt{-g}\Big[-\frac{\Lambda}{8\pi G} + \frac{1}{16\pi G} R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \mathcal{O}(\nabla^6)\Big]. \tag{24}$$

Boundary terms required by the variational problem are understood. The zero-derivative cosmological term is part of the leading action and is not suppressed at large distance. Its observed smallness, if assumed, is a separate hierarchy problem.

Proposition 5.1 (Relative derivative suppression)**.** *Let a background have curvature scale $R_{\mu\nu\rho\sigma} = \mathcal{O}(L^{-2})$ and derivatives $\nabla = \mathcal{O}(L^{-1})$. Under Theorems 3.9 and 3.10, the contribution of a local operator with $2n$ derivatives relative to the Einstein-Hilbert contribution is*

$$\mathcal{O}\big(\epsilon^{2n-2}\big) \tag{25}$$

up to dimensionless bounded Wilson coefficients and field-amplitude factors. This estimate does not apply to nonanalytic terms generated by retained massless fields.

Proof. Each additional pair of derivatives supplies L^{-2}. The coefficient of a $2n$-derivative operator supplies the corresponding power of the short scale ℓ_*^{2n-2} after the common two-derivative normalization is removed. Their ratio is $(\ell_*/L)^{2n-2}$. The coefficient bound prevents inverse powers of ϵ from canceling this ratio. Massless propagators produce nonanalytic momentum dependence, so they cannot be placed in this local Taylor series. □

Theorem 5.2 (Conditional two-derivative metric action)**.** *Let $d > 2$ and assume regular Lorentzian reconstruction, infrared locality, diffeomorphism redundancy, scale separation, controlled Wilson coefficients, and the single-metric spectrum axiom. Suppose also that the pure metric action is parity even and analytic at two derivatives. Then, modulo a boundary term and invertible local field redefinitions, its local zero- and two-derivative part is*

$$\Gamma^{(\le 2)}_{\mathrm{grav}} = \frac{1}{16\pi G} \int_M \mathrm{d}^d x \sqrt{-g}\,(R - 2\Lambda). \tag{26}$$

The theorem does not assert that G or Λ is determined by the information metric.

Proof. Diffeomorphism redundancy restricts local terms to integrals of scalar densities. At zero derivatives, the only pure metric scalar density without additional background structures is $\sqrt{-g}$. At two derivatives, curvature is the covariant tensor built from the metric, and the only parity-even scalar linear in curvature is R, up to a divergence. Locality and analyticity permit the derivative ordering. The spectrum axiom identifies g as the field carrying the nondegenerate long-range spin-two kinetic term, so its coefficient is nonzero. Scale separation and coefficient control place higher-derivative local operators in the remainder. Nothing in these steps computes the coefficients from $\mathcal{G}$. □

The theorem is an effective-action statement, not yet an equation of motion. To infer local Euler-Lagrange equations from state variations, one also needs Theorem 3.4. To couple a conserved matter source consistently, one needs the Ward and anomaly hypotheses in Section 7.

6 First order, second order, and the missing dictionary

Let

$$D_B(t) = D\big(\rho_B(t)\|\sigma_B\big), \qquad \rho_B(0) = \sigma_B. \tag{27}$$

Using $K_B = -\log \sigma_B$ and $\operatorname{Tr} \dot{\rho}_B = 0$ gives

$$\dot{D}_B(0) = \frac{\mathrm{d}}{\mathrm{d}t} \big(\langle K_B\rangle_{\rho(t)} - S(\rho_B(t))\big)_{t=0} = 0. \tag{28}$$

This is the entanglement first law. At second order,

$$\ddot{D}_B(0) = \mathcal{G}_B^{\mathrm{BKM}}(\dot{\rho}_B, \dot{\rho}_B) \geq 0 \tag{29}$$

on the faithful stratum.

Proposition 6.1 (Order separation)**.** *The first-law condition* (14) *fixes the vanishing linear term of relative entropy about the reference state. Positivity* (15) *constrains the leading nonzero quadratic term when the expansion exists. Neither condition implies that a state perturbation determines a local metric perturbation, nor that the metric perturbation has a diffeomorphism gauge redundancy.*

Proof. The two statements concern derivatives of a scalar functional on the state manifold. The required geometric conclusions refer to the support, codomain, and quotient properties of $\mathfrak{R}_\mu$. Those properties are absent from the premises. The preferred-chart construction following Theorem 4.2 satisfies the standard relative-entropy identities while violating diffeomorphism redundancy. An all-to-all reconstruction kernel can satisfy the same identities while violating infrared locality. □

Three dictionaries used in the literature illustrate the missing input. For a ball in a conformal field theory vacuum, the modular Hamiltonian has a local stress-tensor expression obtained by conformal mapping from a Rindler wedge [12, 13]. In holography, the Ryu-Takayanagi area formula and the boundary stress dictionary relate the first law to a bulk constraint [14, 5, 6]. In a local Rindler-horizon construction, Jacobson relates heat flux, horizon temperature, and entropy [8]. In the later small-causal-diamond construction, he assumes vacuum entanglement stationarity at fixed volume and uses a local modular-energy relation for conformal matter [7]. These are related constructions, but they are not instances of one dictionary that has been proved for arbitrary quantum systems.

Definition 6.2 (Entanglement-gravity dictionary)**.** For a class of regions B, an entanglement-gravity dictionary consists of:

1. a map from state perturbations to admissible metric and matter perturbations;
2. a geometric functional whose first variation represents the geometric entropy contribution;
3. a relation between modular energy and the reconstructed matter stress tensor, with region-dependent coefficients;

4. a completeness statement showing that the resulting integral constraints determine the claimed local tensor equation.

Remark 6.3. The higher-curvature small-ball analysis replaces ordinary volume by a generalized volume and yields linearized higher-derivative constraints [15]. It does not generally promote those linearized constraints to the full nonlinear higher-curvature equations. This is one reason not to include higher-curvature conclusions inside Theorem 3.7.

7 Conservation, Ward identities, and anomalies

Assume an infinitesimal diffeomorphism generated by a compactly supported vector field ξ^μ. The field variations are

$$\delta_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}, \qquad \delta_\xi \Phi = \mathcal{L}_\xi \Phi. \tag{30}$$

For a local diffeomorphism-invariant action, integration by parts gives a Noether identity relating the metric Euler-Lagrange tensor to the matter equations. The covariant phase-space treatment of such identities and their boundary terms is developed by Iyer and Wald [18].

Define

$$E_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \Gamma_{\rm grav}}{\delta g^{\mu\nu}}. \tag{31}$$

With (12), the metric equation from stationarity would be $E_{\mu\nu} - T_{\mu\nu} = 0$ in this normalization. Later papers may extract a conventional factor of $8\pi G$ from $E_{\mu\nu}$.

Lemma 7.1 (On-shell Ward consequence)**.** *Assume Theorem 3.6 and the effective matter equations. Then the matter stress tensor defined by* (12) *obeys Theorem 3.5. For a diffeomorphism-invariant gravitational action, its Euler-Lagrange tensor also obeys*

$$\nabla^\mu E_{\mu\nu} = 0 \tag{32}$$

off shell with respect to the metric, modulo the equations of any additional gravitational fields included in Φ*.*

Proof. Vary the effective action under (30). The anomaly-free Ward identity sets the variation to zero. After integration by parts, the coefficient of the arbitrary compactly supported ξ^ν is the divergence of the metric variational derivative plus terms proportional to the Euler-Lagrange derivatives of Φ. On the matter equations, the latter vanish, giving conservation. Applying the same argument to the gravitational action gives (32) with the stated qualification. □

Remark 7.2 (Why the Bianchi identity is not enough). The contracted Bianchi identity gives $\nabla^\mu G_{\mu\nu} = 0$. It does not, by itself, define a matter stress tensor or show that a quantum matter measure is anomaly free. If one first assumes the Einstein equation, the Bianchi identity imposes a consistency condition on its source. That reverse inference is not a derivation of the matter Ward identity.

If a consistent gravitational anomaly is present, the Ward identity takes the form

$$\nabla^\mu T_{\mu\nu} = \mathcal{A}_\nu[g, \Phi], \tag{33}$$

with a local anomaly polynomial descendant $\mathcal{A}_\nu$. The reconstructed metric may remain smooth and local, and relative entropy may remain positive, while the source cannot satisfy the ordinary conservation law. Thus Theorem 3.6 and Theorem 3.5 cannot be inferred from the information axioms.

8 Spectrum and dimension

The phrase "leading metric theory" is incomplete until the long-range fields and the dimension are specified. Effective field theory retains every gapless field compatible with the symmetries. Integrating out a field is justified by a gap and scale separation, not by a desire for a pure metric equation.

8.1 Additional long-range fields

Consider a scalar-tensor action

$$\Gamma[g, \phi] = \int \mathrm{d}^d x \sqrt{-g} \left[\frac{1}{2} F(\phi) R - \frac{1}{2} Z(\phi) (\nabla \phi)^2 - V(\phi) \right]. \tag{34}$$

It is local and diffeomorphism invariant. If ϕ is massless, it contributes at the same long distances as the metric. The Brans-Dicke theory is the classic example [19]. No power of ℓ_*/L removes this mode. The pure Einstein-Hilbert conclusion therefore needs the spectral axiom.

A preferred timelike vector field gives a covariant vector-tensor theory when the vector is itself dynamical. Einstein-aether theory provides an explicit example [20]. The action has diffeomorphism redundancy but its spectrum and characteristic cones are not those of a single metric graviton.

Multiple massless spin-two fields require special care. Local cross-interactions of Pauli-Fierz fields face strong consistency restrictions [21]. Even if interactions force decoupled sectors, the long-range response is not encoded by a single Newton coefficient unless a further spectral projection is made.

8.2 Two dimensions

In $d = 2$,

$$\int_M \sqrt{|g|}\, R \tag{35}$$

is, with the appropriate boundary term, proportional to the Euler characteristic. Its bulk metric variation vanishes identically. Consequently a nonzero coefficient of (35) does not give a propagating Einstein kinetic term. Two-dimensional gravity can be dynamical through a dilaton, a conformal anomaly, or other fields, but then the hypotheses and conclusion differ from a pure metric Einstein theory.

8.3 Three dimensions

In $d = 3$, the Riemann tensor is algebraically determined by the Ricci tensor. Vacuum Einstein gravity has no local graviton polarization, although it has global, boundary, and topological degrees of freedom. The classical dynamics of three-dimensional Einstein gravity and its point-source solutions were analyzed by Deser, Jackiw, and 't Hooft [22]. Thus Theorem 5.2 remains an action classification in three dimensions, but it does not establish the propagating response assumed in a stiffness interpretation.

8.4 Four and higher dimensions

In $d = 4$, the Gauss-Bonnet combination is topological for constant coefficient and does not add a local bulk equation on a closed manifold. Generic curvature-squared terms do add four-derivative corrections. The Lovelock classification describes natural symmetric divergence-free tensors with the stipulated derivative properties [23, 24].

For $d > 4$, higher Lovelock densities need not be topological. They yield second-order field equations despite containing higher powers of curvature. Around a weakly curved background their contributions can still be smaller by powers of curvature relative to the cutoff, but they cannot be excluded merely by saying that the equations contain at most second derivatives. Dimension and coefficient control remain independent assumptions.

Proposition 8.1 (Dimension-sensitive reading of the spectrum axiom)**.** *The assertion "one massless spin-two mode" has the following distinct content:*

1. *in* $d = 2$*, it cannot be realized by the pure Einstein-Hilbert metric action;*
2. *in* $d = 3$*, the Einstein-Hilbert action has no local bulk graviton polarizations;*
3. *in* $d \geq 4$*, it selects a nondegenerate physical spin-two kinetic sector, subject to background and boundary conditions.*

Therefore the dimension axiom cannot be replaced by a dimension-free spectrum slogan.

Proof. The first statement follows from the Gauss-Bonnet theorem in two dimensions. The second follows from the algebraic determination of the Riemann tensor by the Ricci tensor in three dimensions. The third is the usual local polarization statement for a massless spin-two field in dimensions at least four, after quotienting linearized diffeomorphisms. □

9 Dependency theorem

We now state what the axioms establish before any field equation is derived. Let

$$\begin{aligned} &\mathsf{K} = \{R, L, D\}, && \mathsf{E} = \{S, N, Q, M\}, && \\ &\mathsf{W} = \{A, C\}, && \mathsf{I} = \{F, P\}, && \mathsf{V} = \{V\}. \end{aligned} \tag{36}$$

These groupings are mnemonic only. They do not add implications among their members.

Theorem 9.1 (Infrared admissibility and dependency)**.** *Let $\mathfrak{X}_\mu$ be a reconstruction record. Then the following conditional statements hold.*

1. *R+L+D make g a smooth local infrared field with diffeomorphism-related representatives. None of these three properties follows from the existence of the information metric $\mathcal{G}$ alone.*

2. *$S + N + Q + M$, together with the first item, select the Einstein-Hilbert plus cosmological action as the pure metric zero- and two-derivative local action in the domain of Theorem 5.2.*

3. *A and the matter equations imply C. Neither F nor P implies A or C.*

4. *F supplies a first-order state-space identity and P supplies second-order positivity. A gravitational constraint requires an entanglement-gravity dictionary in addition to either statement.*

5. *V is required to pass from stationarity under reconstruction-induced variations to a local metric Euler-Lagrange equation. It is not required merely to write the effective action.*

Proof. The first statement follows directly from the types of the three axioms and from Theorems 4.1 and 4.2. The second is Theorem 5.2 with the power-counting estimate of Theorem 5.1. The third is Theorem 7.1; the countermodels in Section 10 show the stated nonimplications. The fourth follows from Theorem 6.1 and Theorem 6.2. For the fifth, the fundamental lemma of the calculus of variations applies only when the allowed variations span the local test variations modulo gauge. That spanning condition is exactly Theorem 3.4. ☐

Corollary 9.2 (No automatic gravitational gauge principle)**.** *A monotone quantum information metric, relative-entropy positivity, and the entanglement first law do not entail diffeomorphism redundancy of a reconstructed Lorentzian metric.*

Proof. All three informational properties can hold on the faithful state simplex in the preferred-chart example of Section 4, while Theorem 3.3 fails. ☐

Corollary 9.3 (No dimension-free Einstein conclusion)**.** *Locality, diffeomorphism redundancy, conservation, and scale separation do not by themselves imply a propagating Einstein metric sector.*

Proof. Pure Einstein gravity in two dimensions has a topological Einstein-Hilbert term, and in three dimensions it has no local bulk graviton polarization. Scalar-tensor and topological infrared theories provide further counterexamples. ☐

10 Remove-one-axiom countermodels

This section tests necessity in the semantic sense defined in Section 2. A countermodel need not be a microscopic theory of our world. It must show that the remaining predicates do not logically imply the omitted one or the desired conclusion.

10.1 Removing regular Lorentzian reconstruction

Take a finite spin chain with its smooth manifold of faithful density matrices and the Bogoliubov-Kubo-Mori metric. Coarse-grain to its finite set of gapped ground-state sectors. The output is a zero-dimensional label space, not a smooth Lorentzian manifold. Relative entropy is positive and the entanglement first law holds on faithful state families. There is no Lorentzian metric field to which the remaining infrared axioms could apply. The toric-code phase supplies a familiar topological model whose low-energy data do not require a propagating metric mode [25].

10.2 Removing infrared locality

Let M be a smooth Lorentzian manifold and consider

$$\Gamma[g] = \frac{1}{16\pi G}\int \sqrt{-g}\, R + \gamma \int \sqrt{-g}\, R\, \Box^{-1} R. \tag{37}$$

The action is diffeomorphism invariant when $\Box^{-1}$ and its boundary conditions are defined covariantly. It can be anomaly free and its sources can be conserved. If the second term is unsuppressed at the momenta of interest, no local derivative expansion captures the leading response. Thus redundancy, conservation, and a Lorentzian output do not imply Theorem 3.2.

At the reconstruction level, the same point is made by

$$\delta_J g_{\mu\nu}(x) = u_{\mu\nu}(x)\, \mathrm{Tr}(\mathcal{K}_U J), \tag{38}$$

where $u_{\mu\nu}$ has support throughout M. $\mathcal{K}_U \in \mathcal{A}(U)$ is an observable paired with the trace-class tangent J. A perturbation of one microscopic region changes the metric everywhere with no decay scale.

10.3 Removing reconstruction redundancy

The preferred-chart model (22) has a smooth Lorentzian output and may be chosen local in the parameter $X(\lambda)$. Add a fixed nondynamical tensor u_μ to the effective action through

$$\int \sqrt{-g}\, u^\mu u^\nu R_{\mu\nu}. \tag{39}$$

If u is held fixed rather than transformed as a field, active diffeomorphisms do not relate equivalent representatives. The information metric remains monotone and relative entropy remains positive. The omitted redundancy is not recovered.

10.4 Removing variational completeness

Fix a background $\bar{g}$ and let the image of the reconstruction be the one-parameter family

$$g_{\mu\nu}(a) = e^{2a}\bar{g}_{\mu\nu}. \tag{40}$$

The reconstruction is smooth and can respect diffeomorphism redundancy if a is a scalar modulus. Stationarity of an action along this image gives

$$\int_M \sqrt{-g}\, E^{\mu\nu} g_{\mu\nu} = 0. \tag{41}$$

This single integrated trace condition does not imply $E_{\mu\nu} = 0$ pointwise. All other local metric variations are absent from the image.

10.5 Removing matter conservation

Let $J_{\mu\nu}$ be a prescribed smooth symmetric source with $\nabla^\mu J_{\mu\nu} \neq 0$ and write the proposed equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = J_{\mu\nu}. \tag{42}$$

The left side is divergence free, so the equation has no solution on regions where the source divergence is nonzero. The metric action can remain local and diffeomorphism invariant. Failure lies in the source sector. Relative-entropy identities on an unrelated microscopic state family do not repair this incompatibility.

10.6 Removing anomaly freedom

Consider chiral matter in a dimension and representation with a nonvanishing gravitational anomaly. The classical action can be generally covariant, the reconstruction can be smooth and local, and the state relative entropy can be positive. Quantum mechanically the effective action satisfies (33) rather than (13). If no inflow sector cancels $\mathcal{A}_\nu$, the ordinary diffeomorphism constraint is inconsistent with the matter effective action. This countermodel distinguishes Theorem 3.6 from the on-shell statement Theorem 3.5.

10.7 Removing the entanglement first law

The ordinary first law follows for differentiable faithful state paths with the von Neumann entropy. It can fail as a usable hypothesis when the proposed path crosses a rank-changing boundary, so that $K = -\log\sigma$ is unbounded on the new support and the required derivative does not exist. One may nevertheless have a smooth classical spacetime effective action, a local metric sector, diffeomorphism redundancy, conservation, and a controlled spectrum. The effective-action route does not require (14). Thus F is not necessary for infrared effective-action admissibility, although it is necessary for the entanglement route defined in Theorem 6.2.

10.8 Removing relative-entropy positivity

To separate logical roles, replace Umegaki relative entropy in an otherwise fixed reconstruction record by a signed contrast functional

$$\widetilde{D}(\rho\|\sigma) = D(\rho\|\sigma) - 2D(\sigma\|\rho). \tag{43}$$

Its first variation still vanishes on the diagonal whenever both terms are differentiable, so a first-law identity can be written. Its quadratic form is not positive in general. The local action and diffeomorphism structure can be held fixed independently. This formal countermodel shows that a diagonal first-law statement alone does not imply a positive information metric. It is not a proposal to replace quantum relative entropy physically.

10.9 Removing scale separation

Let the microscopic correlation length be of the same order as the curvature radius, $\ell_* \sim L$. Then every term in (24) can contribute at comparable size. The reconstructed output may still be Lorentzian, local in an exact microscopic sense, and diffeomorphism redundant. There is no controlled reason to truncate at two derivatives. The Einstein-Hilbert term can be present without being dominant.

10.10 Removing coefficient control

Keep $\epsilon \ll 1$ but take

$$\alpha_1 \sim \frac{L^2}{G}. \tag{44}$$

On curvature scale L^{-2}, the R^2 variation is then comparable to the Einstein variation. The gap and scale separation remain intact, but the expected derivative suppression is canceled by the Wilson coefficient. This shows why Theorem 3.10 is independent of Theorem 3.9.

10.11 Removing the spectrum assumption

The scalar-tensor action (34) with a massless scalar satisfies locality, diffeomorphism redundancy, anomaly freedom, and scale separation. Its leading field equations include derivatives of $F(\phi)$ and a scalar equation. They cannot be rewritten as a pure Einstein equation with a fixed Newton constant without changing the matter-gravity split or making additional approximations.

As a second example, two decoupled Einstein-Hilbert sectors

$$\Gamma[g, f] = \frac{1}{16\pi G_g} \int \sqrt{-g}\, R[g] + \frac{1}{16\pi G_f} \int \sqrt{-f}\, R[f] \tag{45}$$

have two massless spin-two modes and two diffeomorphism redundancies. They obey the other structural axioms sector by sector but violate the one-metric spectrum hypothesis.

10.12 Removing the dimension domain

Apply the phrase "Einstein-Hilbert gives the leading propagating metric mode" in $d = 2$. The action is topological and its bulk variation vanishes. Apply it in $d = 3$. The action has no local bulk graviton polarization. Alternatively, apply a four-dimensional Lovelock truncation unchanged in $d > 4$. Higher Lovelock tensors may contribute while maintaining second-order equations. These examples satisfy many dimension-blind axioms and fail the intended conclusion.

10.13 Countermodel summary

Omitted axiom	Countermodel	Lost conclusion
R	Gapped finite or topological infrared sector with a faithful state metric	No smooth Lorentzian field
L	Unsuppressed $R\Box^{-1}R$ or all-to-all reconstruction kernel	No local derivative expansion
D	Preferred chart and fixed background tensor	No diffeomorphism quotient
V	One global conformal reconstruction mode	Integrated stationarity only
C	Prescribed source with nonzero divergence	Incompatible local metric equation
A	Uncanceled gravitational anomaly	Anomalous Ward identity
F	Rank-changing nondifferentiable state path	No first-order entanglement route
P	Signed contrast with vanishing diagonal first variation	Indefinite quadratic response
S	$\ell_* \sim L$	No derivative hierarchy
N	$\alpha_1 \sim L^2/G$	Higher curvature remains leading
Q	Massless scalar-tensor or two-metric theory	No pure one-metric leading law
M	Two- or three-dimensional pure Einstein action	No local propagating metric mode

The table is not a list of mutually exclusive physical theories. It is a list of witnesses to nonimplication. Some rows deliberately hold a formal state-space construction fixed while changing the infrared dynamics, because that is precisely what tests whether the information axioms determine the dynamics.

11 Minimal admissibility classes

Different later arguments require different subsets of the axioms. It is useful to name those subsets without treating one as a consequence of another.

Definition 11.1 (Local geometric reconstruction class)**.** The class $\mathfrak{C}_{\mathrm{geom}}$ contains records satisfying

$$R + L + D. \tag{46}$$

Its objects have smooth local Lorentzian representatives modulo diffeomorphisms. No action, conservation law, or entanglement dictionary is part of the definition.

Definition 11.2 (Wilsonian metric class)**.** The class $\mathfrak{C}_{\mathrm{EFT}}$ contains records satisfying

$$R + L + D + S + N + Q + M + A. \tag{47}$$

It admits the local action statement of Theorem 5.2 in its dimension and spectrum domain. Variational completeness is added only when local equations are to be inferred from microscopic stationarity.

Definition 11.3 (Entanglement-constrained reconstruction class)**.** The class $\mathfrak{C}_{\mathrm{ent}}$ contains records in $\mathfrak{C}_{\mathrm{geom}}$ that satisfy $F + P$ and carry an entanglement-gravity dictionary in the sense of Theorem 6.2. The dictionary specifies whether the argument is holographic, based on small causal diamonds, or of another stated type.

Definition 11.4 (Ward-compatible variational class)**.** The class $\mathfrak{C}_{\mathrm{var}}$ contains records in $\mathfrak{C}_{\mathrm{EFT}}$ that satisfy $V + C$ and use boundary conditions for which the variational problem is well posed.

Proposition 11.5 (No inclusion from information data alone)**.** *The presence of $F + P$ does not imply membership in $\mathfrak{C}_{\mathrm{geom}}$, $\mathfrak{C}_{\mathrm{EFT}}$, or $\mathfrak{C}_{\mathrm{var}}$.*

Proof. A faithful finite-dimensional quantum state family satisfies the standard first law and positivity. Choose the reconstruction to be absent, topological, nonlocal, or based on a preferred chart. Each choice violates at least one defining axiom of the named classes while leaving $F + P$ unchanged. □

These classes prevent later papers from citing "the reconstruction assumptions" as an undifferentiated package. An effective-action derivation should cite $\mathfrak{C}_{\mathrm{var}}$. An entanglement derivation should cite $\mathfrak{C}_{\mathrm{ent}}$ and then name its dictionary. A comparison of the routes must assume that both structures are present on the same reconstruction record.

12 Limitations and open mathematical tasks

The framework begins after a reconstruction map has been proposed. It does not construct M or g from an arbitrary algebra and state. Establishing such a construction for a nonholographic microscopic model remains a model-dependent problem.

The locality axiom is formulated by decay of response kernels and by a Wilsonian expansion. For continuum algebraic quantum field theory, a more complete formulation should use inclusions of von Neumann algebras, modular data, and a precise topology on reconstruction maps. For lattice systems, Lieb-Robinson estimates give a natural microscopic control of quasi-locality, but an additional argument is needed to transfer that control to a Lorentzian reconstruction.

The redundancy axiom presupposes an equivalence relation whose observables are known. In gauge-gravity duality, code-subspace reconstruction provides examples of redundant encoding. Outside that domain, finding separating observables and proving the faithful intertwiner (20) are substantial tasks.

The spectrum axiom is stated at the level needed for effective field theory. A microscopic proof would have to identify poles or long-distance representations, show that the spin-two kinetic form is nondegenerate after gauge quotient, exclude ghosts, and bound mixing with other gapless sectors. The consistency arguments of massless spin-two gauge theory explain why universal coupling is natural once such a mode exists [26, 27]. They do not prove that the mode exists in a given microscopic theory.

Anomaly freedom must be checked for the complete field content, including boundary modes and inflow sectors. A low-energy truncation that discards an anomaly-canceling mode can make the remaining description inconsistent even when the microscopic theory is sound. The correct admissible record then includes the compensating topological term or boundary sector.

Finally, Haskell validation of finite axiom records can check dependency bookkeeping and counterexample coverage. It cannot prove the continuum locality, Ward, spectrum, or reconstruction hypotheses. Those are mathematical and physical statements about the intended models.

13 Conclusion

A reconstructed Lorentzian tensor becomes a candidate infrared metric field only after its locality and redundancy properties are specified. It becomes the variable of a controlled metric effective theory only after scale, coefficient, spectrum, dimension, and anomaly assumptions are added. Conservation follows from the appropriate anomaly-free Ward identity together with the matter equations, rather than from information geometry. The entanglement first law and relative-entropy positivity remain separate inputs at first and second order, and their gravitational use requires an explicit dictionary.

The remove-one-axiom constructions show why each separation matters. They also set a clear boundary for subsequent variational arguments. Within the Wilsonian metric class, the Einstein-Hilbert and cosmological terms form the zero- and two-derivative pure metric action in dimensions greater than two. Turning its stationarity into a local equation requires variational completeness, and claiming a propagating graviton requires the dimension-sensitive spectrum assumption. No step identifies diffeomorphism redundancy as an automatic consequence of a quantum information metric.

A A formal dependency algebra

Let the universe of atomic hypotheses be

$$\mathbb{A} = \{R, L, D, V, C, A, F, P, S, N, Q, M\}. \tag{48}$$

For a conclusion X, let $\mathrm{Dep}(X)$ denote one sufficient set of atomic hypotheses established in this paper. The principal entries are

$$\begin{aligned}
\mathrm{Dep}(\text{local Lorentzian representative}) &= \{R, L\}, && (49)\\
\mathrm{Dep}(\text{diffeomorphism quotient}) &= \{R, D\}, && (50)\\
\mathrm{Dep}(\text{two-derivative metric action}) &= \{R, L, D, S, N, Q, M\}, && (51)\\
\mathrm{Dep}(\text{on-shell stress conservation}) &= \{A, \text{matter equations}\}, && (52)\\
\mathrm{Dep}(\text{local equation from stationarity}) &= \{V, \text{stationarity}\}, && (53)\\
\mathrm{Dep}(\text{entanglement constraint}) &= \{F, \text{entanglement-gravity dictionary}\}, && (54)\\
\mathrm{Dep}(\text{positive canonical response}) &= \{P, \text{canonical-energy dictionary}\}. && (55)
\end{aligned}$$

The word "sufficient" is important. There can be alternative constructions with different premises. For example, a discrete causal structure could yield an infrared locality notion without first presenting the smooth response kernel in Theorem 3.2. Such an alternative must still establish the property used in a later theorem.

Define a validation predicate

$$\mathrm{valid}(H, X) \iff \mathrm{Dep}(X) \subseteq H, \tag{56}$$

where $H \subseteq \mathbb{A}$ is the set of checked hypotheses. This predicate is monotone:

Lemma A.1 (Monotonicity of dependency validation)**.** *If $H \subseteq H'$ and $\mathrm{valid}(H, X)$, then $\mathrm{valid}(H', X)$.*

Proof. From $\mathrm{Dep}(X) \subseteq H$ and $H \subseteq H'$, transitivity of set inclusion gives $\mathrm{Dep}(X) \subseteq H'$. □

The converse fails. A larger record may validate X even when a smaller one does not. Remove-one-axiom tests compute whether a particular atom belongs to the chosen sufficient set:

$$a \in \mathrm{Dep}(X) \implies \neg\,\mathrm{valid}\big(\mathrm{Dep}(X) \setminus \{a\}, X\big). \tag{57}$$

This bookkeeping identity is not a proof of physical necessity. Physical necessity is supported by the countermodels in Section 10.

B Power counting with explicit dimensions

In units $\hbar = c = 1$, the action is dimensionless and

$$[x] = -1, \qquad [R] = 2, \qquad [G] = 2 - d. \tag{58}$$

Write the local pure metric Lagrangian as

$$\mathcal{L}_{\mathrm{grav}} = M_*^d \sum_{n \geq 0} c_n \frac{\mathcal{O}_{2n}}{M_*^{2n}}, \tag{59}$$

where $\mathcal{O}_{2n}$ denotes a scalar with $2n$ derivatives and the c_n are dimensionless after a basis is chosen. For $R \sim L^{-2}$,

$$\frac{c_n M_*^{d-2n} L^{-2n}}{c_1 M_*^{d-2} L^{-2}} = \frac{c_n}{c_1}(M_* L)^{-2n+2}. \tag{60}$$

Taking $M_* \sim \ell_*^{-1}$ reproduces (25).

This calculation makes three caveats explicit. First, c_0 is multiplied by $(M_* L)^2$ relative to the curvature term, so the cosmological contribution is not infrared suppressed. Second, if c_n/c_1 grows with $M_* L$, the derivative estimate fails. Third, amplitudes can introduce additional scales. For example, a scalar background with $\nabla\phi$ much larger than its nominal infrared estimate can invalidate a mixed metric-scalar expansion even when the curvature is small.

Massless loops produce terms schematically of the form

$$\int \sqrt{-g}\, R \log\left(\frac{-\Box}{\mu^2}\right) R, \tag{61}$$

which have no Taylor expansion around zero momentum. Their coefficients and long-distance effects are treated by the quantum effective field theory rather than absorbed into local c_n.

C Finite records and executable validation

The companion Haskell modules represent each axiom as a distinct field of a typed record. They define conclusions by explicit dependency sets and check remove-one-axiom examples. The representation is intentionally finite and propositional. For example,

$$\text{requires}(\texttt{TwoDerivativeMetricAction}) = \{R, L, D, S, N, Q, M\}. \tag{62}$$

Deleting D invalidates that conclusion but does not change the truth value stored for F or P.

This executable model checks two useful facts. Every named conclusion has a declared set of requirements. Every atomic requirement used in the main action conclusion has a remove-one record that fails validation for that conclusion. It does not establish that a continuum reconstruction is local or anomaly free. Nor do finite Boolean fields capture approximate validity, scale-dependent error bars, or relations among different definitions of locality.

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