Microscopic Distinguishability and the Data of Geometric Reconstruction

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Abstract

This paper isolates the microscopic input needed by proposals that reconstruct geometry from quantum distinguishability. A smooth family of faithful states carries many contractive Riemannian metrics, not one undifferentiated quantum Fisher metric. The mixed Hessian of Umegaki relative entropy, with a necessary minus sign, gives the Bogoliubov-Kubo-Mori metric. The symmetric logarithmic derivative Fisher tensor is a different Petz monotone metric, while the Bures tensor is one quarter of it under the conventions used here on a smooth constant-rank stratum. At rank-changing points, parameter Fisher information and the quadratic coefficient of Bures distance need not agree by continuous extension. We formulate a reconstruction interface for a local net of state families, a chosen monotone metric, coarse-graining channels, localization maps, probe maps, and independent causal data. The interface separates what follows from information geometry from what must be supplied to obtain a smooth Lorentzian manifold. Elementary propositions establish the mixed-Hessian sign, monotonicity consequences, identifiability criteria, the impossibility of obtaining Lorentzian signature by a bare pullback of a positive metric, and a conditional Lorentzianization construction from a normalized clock covector. Finite classical and qubit examples show both the useful content and the limitations of the definitions. No existence theorem for spacetime reconstruction from an arbitrary quantum system is assumed or claimed.

Contents

1 Introduction

Information geometry begins with an operationally meaningful question. How rapidly can nearby states be distinguished when the state label is varied? For classical probability distributions, the local answer is encoded by the Fisher metric. For density operators, noncommutativity permits a family of inequivalent contractive metrics. This difference is not a minor convention. It determines which microscopic response coefficients are being compared and which data-processing statements are available.

Proposals relating quantum information to spacetime often move too quickly between three distinct constructions. The first is a positive metric on a manifold of states. The second is a locality structure that associates states and observables with subsystems. The third is a Lorentzian metric on another manifold whose points, causal directions, and fields must themselves be identified. A positive state-space metric cannot by pullback alone acquire Lorentzian signature. Consequently, a reconstruction proposal needs more than a choice of distinguishability measure.

The purpose of this paper is to state that additional data as an interface rather than hide it in an informal arrow. The interface is deliberately conditional. It records what a reconstruction must accept, what it may return, and which compatibility properties can be tested. It does not assert that a generic spin system, matrix algebra, field theory, or tensor network satisfies those properties.

There are four technical points that guide the discussion. First, the sign in a mixed Hessian of a two-point divergence is fixed by positivity along relative displacement. For

Umegaki relative entropy the metric is

$$g_{ab}^{\mathrm{BKM}}(\lambda) = -\frac{\partial^2}{\partial\lambda^a\partial\lambda'^b} D\big(\rho(\lambda)\|\rho(\lambda')\big)\Big|_{\lambda'=\lambda}. \tag{1}$$

Without the minus sign, the resulting tensor is negative in the elementary classical example.

Second, the BKM metric in equation (1) is not the Bures metric. It is also not the symmetric logarithmic derivative Fisher tensor. All three belong to the Petz family after normalizations are fixed, but their operator means differ. The Bures and SLD tensors are proportional on a smooth constant-rank stratum under common conventions. The BKM tensor generally is not proportional to either of them in noncommuting directions.

Third, rank matters. The manifold of faithful density operators is smooth, and the standard operator formulas are unambiguous there. The boundary of state space is stratified by rank. A parametrized curve that creates or removes a zero eigenvalue may have a discontinuous SLD Fisher tensor even when its Bures distance has a well-defined quadratic expansion. We give an explicit two-level example.

Fourth, a reconstruction map must distinguish input hypotheses from output consequences. Locality, smoothness, identifiability, coarse-graining compatibility, and causal structure are independent requirements. The metric data extracted from distinguishability can constrain a reconstruction only after maps between microscopic variations and candidate geometric variations have been supplied.

The main results are modest but exact. Theorem 2.3 proves the sign relation for a smooth divergence. Theorem 2.4 computes the classical relative-entropy Hessian. Theorem 4.8 exhibits a noncommuting qubit direction in which BKM and SLD differ. Theorem 8.1 proves that a bare information-metric pullback cannot be Lorentzian. Theorem 8.3 states a sufficient construction when an independent normalized clock covector is provided. Theorem 8.6 gives the corresponding injectivity test. The remaining statements describe stability under localization and coarse-graining.

The paper is organized as follows. Section 2 fixes divergence conventions using finite probability spaces. Section 3 defines the faithful quantum state manifold and Umegaki relative entropy. Section 4 separates BKM, SLD, Bures, and general Petz metrics. Section 5 analyzes rank-changing curves. Section 6 introduces local nets and coarse-graining. Section 7 defines the reconstruction interface. Section 8 proves consequences of its explicit hypotheses. Section 9 gives finite models and counterexamples. Section 11 records the boundary of the construction.

2 Classical distinguishability as a benchmark

The classical simplex provides a reference point because the standard regularity assumptions select a unique monotone metric up to scale. It also fixes the sign convention before noncommutative operator calculus is introduced.

2.1 The open simplex and its tangent space

Let $X = \{1, \dots, n\}$ be a finite sample space. The open probability simplex is

$$\Delta_n^\circ = \left\{ p = (p_1, \dots, p_n) \in \mathbb{R}^n : p_i > 0, \sum_{i=1}^n p_i = 1 \right\}. \tag{2}$$

Its tangent space at p is canonically identified with

$$T_p\Delta_n^\circ = \left\{ u \in \mathbb{R}^n : \sum_{i=1}^n u_i = 0 \right\}. \tag{3}$$

Definition 2.1 (Classical relative entropy)**.** For $p, q \in \Delta_n^\circ$, the Kullback-Leibler divergence is

$$D_{\mathrm{KL}}(p\|q) = \sum_{i=1}^n p_i \log \frac{p_i}{q_i}. \tag{4}$$

This divergence is nonnegative and vanishes exactly on the diagonal. It is not symmetric and is not a distance. Its local quadratic term is nevertheless symmetric.

Definition 2.2 (Fisher metric)**.** For $u, v \in T_p\Delta_n^\circ$, define

$$g_p^{\mathrm{F}}(u, v) = \sum_{i=1}^n \frac{u_i v_i}{p_i}. \tag{5}$$

Proposition 2.3 (The mixed-Hessian sign)**.** *Let $D(\theta, \theta')$ be a C^2 two-point divergence on a parameter domain. Assume $D(\theta, \theta) = 0$ and that the first derivatives vanish on the diagonal. If*

$$D(\theta, \theta + \delta\theta) = \frac{1}{2} g_{ab}(\theta)\delta\theta^a\delta\theta^b + o(\|\delta\theta\|^2), \tag{6}$$

then

$$g_{ab}(\theta) = \partial_{a'}\partial_{b'} D(\theta, \theta')|_{\theta'=\theta} = -\, \partial_a\partial_{b'} D(\theta, \theta')|_{\theta'=\theta}\,. \tag{7}$$

Proof. The first equality is the Taylor coefficient in the second argument. By hypothesis, the primed gradient vanishes at every point of the diagonal. In particular, the function

$$F_b(\theta) = \partial_{b'} D(\theta, \theta')|_{\theta'=\theta} \tag{8}$$

vanishes identically. Take its total derivative along the diagonal in the θ^a direction. The chain rule gives

$$\partial_a\partial_{b'} D + \partial_{a'}\partial_{b'} D|_{\theta'=\theta} = 0. \tag{9}$$

This is the second equality in equation (7). □

Proposition 2.4 (Relative entropy gives Fisher information)**.** *For smooth $p(\lambda) \in \Delta_n^\circ$,*

$$-\, \partial_a\partial_{b'} D_{\mathrm{KL}}\big(p(\lambda)\|p(\lambda')\big)\big|_{\lambda'=\lambda} = \sum_i \frac{\partial_a p_i\, \partial_b p_i}{p_i}. \tag{10}$$

Proof. Differentiate the second argument first:

$$\partial_{b'} D_{\mathrm{KL}} = -\sum_i p_i(\lambda) \frac{\partial_{b'} p_i(\lambda')}{p_i(\lambda')}. \tag{11}$$

A derivative in the first argument acts only on $p_i(\lambda)$. Setting $\lambda' = \lambda$ yields the negative of the displayed Fisher tensor. The tangent constraint removes the first-order term because $\sum_i \partial_a p_i = 0$. $\square$

2.2 Stochastic coarse-graining

A finite stochastic channel is a matrix $K(y \mid i)$ with nonnegative entries and $\sum_y K(y \mid i) = 1$. It maps p and u to

$$(Kp)_y = \sum_i K(y \mid i) p_i, \qquad (Ku)_y = \sum_i K(y \mid i) u_i. \tag{12}$$

Proposition 2.5 (Classical metric contraction)**.** *For every stochastic channel K,*

$$g^{\mathrm{F}}_{Kp}(Ku, Ku) \le g^{\mathrm{F}}_p(u, u). \tag{13}$$

Proof. For each y with $(Kp)_y > 0$, apply Cauchy-Schwarz to

$$(Ku)_y = \sum_i \sqrt{K(y \mid i) p_i} \frac{\sqrt{K(y \mid i)} u_i}{\sqrt{p_i}}. \tag{14}$$

This gives

$$\frac{(Ku)_y^2}{(Kp)_y} \le \sum_i K(y \mid i) \frac{u_i^2}{p_i}. \tag{15}$$

Summing over y and using $\sum_y K(y \mid i) = 1$ proves equation (13). $\square$

Equality can hold when the channel is sufficient for the tangent direction under consideration. Strict inequality means that the coarse description has discarded locally distinguishable information. The inequality by itself says nothing about spatial locality or Lorentzian causal structure.

3 Faithful quantum state families

Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space. Write $\mathsf{Herm}(\mathcal{H})$ for Hermitian operators and $\mathsf{Pos}(\mathcal{H})$ for positive operators. The faithful density operators form

$$\mathcal{D}_+(\mathcal{H}) = \{\rho \in \mathsf{Pos}(\mathcal{H}) : \rho > 0,\ \operatorname{Tr} \rho = 1\}. \tag{16}$$

This is a smooth manifold whose tangent space is

$$T_\rho \mathcal{D}_+(\mathcal{H}) = \{X \in \mathsf{Herm}(\mathcal{H}) : \operatorname{Tr} X = 0\}. \tag{17}$$

Definition 3.1 (Smooth faithful state family)**.** A smooth faithful state family is a smooth map

$$\rho : \Theta \longrightarrow \mathcal{D}_+(\mathcal{H}) \tag{18}$$

from a finite-dimensional parameter manifold Θ. For $v \in T_\lambda\Theta$, its represented tangent operator is

$$X_v = (\mathrm{d}\rho)_\lambda(v). \tag{19}$$

The map $\mathrm{d}\rho$ need not be injective. A redundant parameter direction then produces the zero tangent operator and has zero information length. This elementary fact becomes important when identifiability is discussed later.

Definition 3.2 (Umegaki relative entropy)**.** For $\rho, \sigma \in \mathcal{D}_+(\mathcal{H})$, define

$$D(\rho\|\sigma) = \mathrm{Tr}\big[\rho(\log\rho - \log\sigma)\big]. \tag{20}$$

Umegaki introduced this noncommutative extension of classical relative entropy in [1]. It is nonnegative and vanishes exactly when $\rho = \sigma$. Its monotonicity under quantum channels was established in finite and operator-algebraic settings by Lindblad and related work [2, 3, 4].

The derivative of the operator logarithm is not multiplication by ρ^{-1} unless the tangent commutes with ρ. For $A > 0$, the Fréchet derivative is

$$(\mathrm{d}\log)_A(Y) = \int_0^\infty (A + sI)^{-1} Y (A + sI)^{-1}\,\mathrm{d}s. \tag{21}$$

Equivalently, define the Kubo-Mori operator

$$\Omega_A(Z) = \int_0^1 A^t Z A^{1-t}\,\mathrm{d}t. \tag{22}$$

On a faithful state, $(\mathrm{d}\log)_A = \Omega_A^{-1}$.

Lemma 3.3 (Eigenbasis formula)**.** *Let $A = \sum_i a_i |i\rangle\langle i|$ with $a_i > 0$. Then*

$$\big[(\mathrm{d}\log)_A(Y)\big]_{ij} = c_{\mathrm{BKM}}(a_i, a_j) Y_{ij}, \tag{23}$$

where

$$c_{\mathrm{BKM}}(x,y) = \begin{cases} \dfrac{\log x - \log y}{x - y}, & x \neq y, \\ \dfrac{1}{x}, & x = y. \end{cases} \tag{24}$$

Proof. Insert the spectral resolution of A into equation (21). The (i, j) entry is multiplied by

$$\int_0^\infty \frac{\mathrm{d}s}{(a_i + s)(a_j + s)}. \tag{25}$$

Partial fractions give equation (24) when $a_i \neq a_j$. The coincident limit is $1/a_i$. □

4 Four quantum information metrics that must be separated

We now define the four notions used in the rest of the paper. The first is a specific contrast-function Hessian. The second is a logarithmic-derivative Fisher tensor. The third is the infinitesimal form of a distance. The fourth is the full classified family of monotone metrics.

4.1 The BKM metric

Definition 4.1 (BKM metric)**.** For a faithful density operator ρ and traceless Hermitian X, Y, define

$$g_\rho^{\mathrm{BKM}}(X,Y) = \mathrm{Tr}\left[X\,\Omega_\rho^{-1}(Y)\right] = \mathrm{Tr}\left[X(\mathrm{d}\log)_\rho(Y)\right]. \tag{26}$$

The name Bogoliubov-Kubo-Mori is used because the inverse logarithmic mean appears in linear response and equilibrium statistical mechanics. The same metric is also called the canonical correlation metric in parts of the literature.

Proposition 4.2 (Relative-entropy Hessian)**.** *Let $\rho(\lambda)$ be a smooth faithful family. Then*

$$g_{ab}^{\mathrm{BKM}}(\lambda) = -\,\partial_a\partial_{b'} D\big(\rho(\lambda)\|\rho(\lambda')\big)\big|_{\lambda'=\lambda} \tag{27}$$

and $g_{ab}^{\mathrm{BKM}} = g_\rho^{\mathrm{BKM}}(\partial_a\rho, \partial_b\rho)$.

Proof. Differentiate equation (20) in the second state. For $Y = \partial_{b'}\rho(\lambda')$,

$$\partial_{b'} D(\rho\|\rho') = -\,\mathrm{Tr}\left[\rho(\mathrm{d}\log)_{\rho'}(Y)\right]. \tag{28}$$

At $\rho' = \rho$, the trace equals $-\,\mathrm{Tr}\,Y = 0$, as required at the diagonal. A derivative in the first state gives

$$-\,\mathrm{Tr}\left[X(\mathrm{d}\log)_\rho(Y)\right]. \tag{29}$$

The minus sign in equation (27) therefore produces equation (26). Symmetry follows from the eigenbasis formula in theorem 3.3. Positivity follows because $c_{\mathrm{BKM}}(a_i, a_j) > 0$ for positive eigenvalues. □

When $[X,\rho] = [Y,\rho] = 0$, the formula reduces to

$$g_\rho^{\mathrm{BKM}}(X,Y) = \sum_i \frac{X_{ii}Y_{ii}}{\rho_i}. \tag{30}$$

Thus every quantum metric considered below restricts to the classical Fisher metric on a commuting submanifold, after the stated normalization is applied.

4.2 The SLD Fisher tensor

Definition 4.3 (Symmetric logarithmic derivative)**.** For $X \in T_\rho\mathcal{D}_+(\mathcal{H})$, the symmetric logarithmic derivative L_X is the unique Hermitian solution of

$$X = \frac{1}{2}(\rho L_X + L_X\rho). \tag{31}$$

Faithfulness makes the superoperator $(L_\rho + R_\rho)/2$ invertible. Here $L_\rho(A) = \rho A$ and $R_\rho(A) = A\rho$.

Definition 4.4 (SLD Fisher tensor)**.** The SLD Fisher tensor is

$$g_\rho^{\mathrm{SLD}}(X, Y) = \frac{1}{2}\operatorname{Tr}\left[\rho(L_X L_Y + L_Y L_X)\right]. \tag{32}$$

In an eigenbasis of ρ this is

$$g_\rho^{\mathrm{SLD}}(X, Y) = \sum_{i,j} \frac{2}{\rho_i + \rho_j} X_{ij} Y_{ji}. \tag{33}$$

For $X = Y$, the expression is real and nonnegative. It is the quantum Fisher information commonly used in local parameter estimation [6, 7].

4.3 Bures distance and its infinitesimal tensor

For density operators ρ and σ, define the root fidelity

$$F_{1/2}(\rho, \sigma) = \operatorname{Tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}. \tag{34}$$

Our squared Bures distance convention is

$$d_{\mathrm{Bu}}^2(\rho, \sigma) = 2\big(1 - F_{1/2}(\rho, \sigma)\big). \tag{35}$$

This convention traces to Bures's distance on positive functionals [5]. A rescaled convention sometimes uses $1 - F_{1/2}$ without the leading factor of two; every Bures tensor quoted in that convention is correspondingly smaller by a factor of two.

Definition 4.5 (Bures metric on a constant-rank stratum)**.** For a smooth curve $\rho(t)$ of constant rank near $t = 0$, define g^{Bu} by

$$d_{\mathrm{Bu}}^2(\rho(0), \rho(t)) = g_{\rho(0)}^{\mathrm{Bu}}(\dot\rho(0), \dot\rho(0))t^2 + o(t^2). \tag{36}$$

The bilinear tensor is obtained by polarization.

On the faithful state manifold, direct differentiation of fidelity gives

$$g_\rho^{\mathrm{Bu}}(X, Y) = \frac{1}{4} g_\rho^{\mathrm{SLD}}(X, Y). \tag{37}$$

Some authors multiply the Bures tensor by four and call the result the Bures quantum Fisher information. Others call equation (32) the quantum Fisher information and reserve Bures metric for equation (37). We retain the factor in equation (37) throughout.

4.4 Petz monotone metrics

The phrase "a Petz metric" refers to a family, not to a single tensor. Let $f : (0, \infty) \to (0, \infty)$ be operator monotone, normalized by $f(1) = 1$, and symmetric in the sense

$$f(t) = t f(t^{-1}). \tag{38}$$

Define the Morozova-Chentsov kernel

$$c_f(x, y) = \frac{1}{y f(x/y)}. \tag{39}$$

Definition 4.6 (Petz monotone metric)**.** The metric associated with f is

$$g^f_\rho(X, Y) = \mathrm{Tr}\left[X\, c_f(L_\rho, R_\rho)(Y)\right]. \tag{40}$$

Petz and Sudár classified monotone Riemannian metrics on faithful matrix state spaces by such functions [8]. Petz further related monotone metrics to contrast functions and generalized covariance constructions [9]. The classification assumes regularity and a faithful-state manifold. It does not identify all metrics with one another.

Two functions are central here:

$$f_{\mathrm{SLD}}(t) = \frac{1+t}{2}, \qquad c_{\mathrm{SLD}}(x, y) = \frac{2}{x+y}, \tag{41}$$

$$f_{\mathrm{BKM}}(t) = \frac{t-1}{\log t}, \qquad c_{\mathrm{BKM}}(x, y) = \frac{\log x - \log y}{x - y}. \tag{42}$$

The coincident values are defined by continuity. The function in equation (41) yields the SLD tensor. The function in equation (42) yields the relative-entropy Hessian. The Bures tensor is one quarter of the SLD tensor under equation (35).

Theorem 4.7 (Contractivity of a Petz metric)**.** *Let Φ be a completely positive trace-preserving map and let g^f be a Petz monotone metric. For faithful ρ and tangent X, with faithful $\Phi(\rho)$,*

$$g^f_{\Phi(\rho)}\big(\Phi(X), \Phi(X)\big) \le g^f_\rho(X, X). \tag{43}$$

Proof. This is the defining monotonicity property in the Petz classification. In finite dimensions it follows from the operator monotonicity of f, the associated transformer inequality for the positive superoperator mean, and a Stinespring representation of Φ. The complete proof is given in [8, 9]. The hypotheses stated here exclude support singularities, which require limits or generalized inverses. □

Proposition 4.8 (BKM and SLD are distinct)**.** *Let*

$$\rho = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix}, \qquad 0 < p < 1. \tag{44}$$

If $p \neq \frac{1}{2}$ and $x \neq 0$, then

$$g^{\mathrm{BKM}}_\rho(X, X) = 2|x|^2 \frac{\log p - \log(1-p)}{2p - 1}, \tag{45}$$

$$g^{\mathrm{SLD}}_\rho(X, X) = 4|x|^2. \tag{46}$$

The ratio depends on p, so the tensors are not related by a global constant.

Table 1: Conventions used for the principal information metrics.

Object	Defining datum	Kernel on faithful states	Relation
BKM	Hessian of $D(\rho\|\sigma)$	$(\log x - \log y)/(x-y)$	Petz metric
SLD Fisher	Symmetric logarithmic derivative	$2/(x+y)$	Petz metric
Bures	Quadratic term of equation (35)	$1/[2(x+y)]$	$g^{\mathrm{Bu}} = g^{\mathrm{SLD}}/4$
Petz f	Symmetric operator-monotone f	$1/[yf(x/y)]$	Classified family

Proof. Only the $(1,2)$ and $(2,1)$ matrix entries contribute. Equation equation (24) gives the same symmetric coefficient for both entries and yields equation (45). Equation equation (33) has denominator $p+(1-p)=1$ for each off-diagonal entry, which yields equation (46). The logarithmic divided difference varies with p. □

Remark 4.9. On commuting directions, both equation (26) and equation (32) reduce to classical Fisher information. The distinction appears in directions that rotate eigenvectors as well as in more general noncommuting variations.

5 Rank strata and boundary effects

The faithful manifold $\mathcal{D}_+(\mathcal{H})$ is only the top stratum of the compact convex state space. For $1 \le r \le \dim \mathcal{H}$, the fixed-rank set

$$\mathcal{D}_r(\mathcal{H}) = \{\rho \ge 0 : \operatorname{Tr}\rho = 1,\ \operatorname{rank}\rho = r\} \tag{47}$$

is a smooth stratum, but strata of different rank meet at singular boundaries. Formulae using ρ^{-1} or $\log\rho$ cannot be applied there without specifying a support convention.

For a nonfaithful state, an SLD need not be unique outside the support. The quadratic form is commonly defined using the spectral formula

$$F_{\mathrm{SLD}}(\rho, X) = 2 \sum_{i,j:\, p_i+p_j>0} \frac{|X_{ij}|^2}{p_i+p_j}. \tag{48}$$

This definition is pointwise. It need not equal the limit of the SLD Fisher information along a rank-changing parametrized family.

Example 5.1 (A rank-changing qubit)**.** For $|t|<1$, let

$$\rho(t) = \begin{pmatrix} t^2 & 0 \\ 0 & 1-t^2 \end{pmatrix}. \tag{49}$$

For $t \ne 0$, the family is faithful and commuting. Its SLD Fisher information is the classical expression

$$F_{\mathrm{SLD}}(t) = \frac{(2t)^2}{t^2} + \frac{(-2t)^2}{1-t^2} = \frac{4}{1-t^2}. \tag{50}$$

Hence $\lim_{t\to 0, t\neq 0} F_{\mathrm{SLD}}(t) = 4$. At $t = 0$, however, $\dot{\rho}(0) = 0$. The parametrized curve is not immersed at that point, so every pointwise bilinear tensor applied to its tangent must vanish there. The pointwise formula equation (48) gives $F_{\mathrm{SLD}}(0) = 0$.

The Bures distance from the endpoint is

$$d^2_{\mathrm{Bu}}(\rho(0), \rho(t)) = 2\left(1 - \sqrt{1 - t^2}\right) = t^2 + O(t^4). \tag{51}$$

Its quadratic coefficient is therefore 1. This is one quarter of the limiting value 4, but not one quarter of the pointwise value 0.

The example displays the precise caveat. The identity $g^{\mathrm{Bu}} = g^{\mathrm{SLD}}/4$ is a tensor identity on a smooth constant-rank stratum. At a rank-changing parameter value, one must state whether the object is a pointwise SLD expression, a limit from a stratum, or the second-order coefficient of a distance along the chosen curve. Zhou and Jiang analyze this distinction and the correction terms that arise at rank changes [12].

Lemma 5.2 (Constant rank removes the elementary discontinuity)**.** *Let $\rho(t)$ be C^2 and of constant rank on a neighborhood of* 0*. Assume its nonzero eigenvalues are bounded below there. Then the spectral coefficients in equation* (48) *vary continuously and the local Bures tensor equals one quarter of the SLD tensor on that stratum.*

Proof. The spectral projection onto the support is smooth under the gap assumption. Every denominator $p_i + p_j$ retained in equation (48) remains uniformly positive when at least one index lies in the support. Functional calculus and the fidelity expansion can therefore be differentiated within the stratum. The standard faithful-state calculation applies on the moving support and gives the stated factor. □

The BKM metric has a stronger boundary singularity because $c_{\mathrm{BKM}}(x, x) = 1/x$. It is naturally defined on faithful states. Limits toward the boundary may diverge, remain finite along specially vanishing tangents, or depend on how the support is approached. A reconstruction that uses BKM data must either remain in a faithful sector or include an explicit boundary prescription.

6 Local state data and coarse-graining

A single density matrix has no intrinsic notion of spatial adjacency. Locality must be supplied by relations among algebras, subsystems, or channels. The algebraic language is convenient because it does not presuppose a background metric.

6.1 Nets over a microscopic region poset

Let $(\mathsf{P}, \subseteq)$ be a poset whose elements represent microscopic regions. It may be the set of subgraphs in a lattice, intervals in a causal order, or an abstract family of subsystems.

Definition 6.1 (Finite local net)**.** A finite local net assigns a unital finite-dimensional C^*-algebra $\mathcal{A}(U)$ to every $U \in \mathsf{P}$ and an injective unital $*$-homomorphism

$$\iota_{UV} : \mathcal{A}(U) \longrightarrow \mathcal{A}(V) \tag{52}$$

whenever $U \subseteq V$. The assignments obey $\iota_{UU} = \mathrm{id}$ and $\iota_{VW} \circ \iota_{UV} = \iota_{UW}$.

This is the finite counterpart of the isotony axiom in algebraic quantum field theory [14]. Additional axioms such as commutativity for disjoint regions, duality, or covariance are not included automatically.

Definition 6.2 (Compatible state family)**.** A compatible family of states is a collection $\rho_U(\lambda)$ such that for $U \subseteq V$,

$$\rho_U(\lambda) = \rho_V(\lambda) \circ \iota_{UV}. \tag{53}$$

In matrix notation this is the appropriate partial trace or channel restriction.

Each region may carry its own tangent metric

$$\mathcal{G}_{U,\lambda}(v, w) = g^f_{\rho_U(\lambda)}\big(\partial_v \rho_U, \partial_w \rho_U\big). \tag{54}$$

If restriction is a quantum channel, theorem 4.7 gives

$$\mathcal{G}_{U,\lambda}(v, v) \leq \mathcal{G}_{V,\lambda}(v, v) \qquad (U \subseteq V). \tag{55}$$

The inequality expresses loss of distinguishability under restriction. It does not say that the difference is additive over regions. Quantum correlations generally obstruct such additivity.

6.2 Scale-indexed channels

Let μ label a coarse-graining scale. For $\mu \preceq \nu$, suppose there is a channel

$$\mathcal{C}_{\nu\leftarrow\mu,U} : \mathsf{S}(\mathcal{A}_\mu(U)) \longrightarrow \mathsf{S}(\mathcal{A}_\nu(U)) \tag{56}$$

and that channels compose consistently across scales. The notation takes larger ν to mean a coarser description, but no numerical convention is essential.

For a smooth state family, the channel also maps tangent operators:

$$X_{\nu,U} = \mathcal{C}_{\nu\leftarrow\mu,U}(X_{\mu,U}). \tag{57}$$

A Petz metric satisfies

$$\mathcal{G}_{\nu,U}(X_{\nu,U}, X_{\nu,U}) \leq \mathcal{G}_{\mu,U}(X_{\mu,U}, X_{\mu,U}). \tag{58}$$

Definition 6.3 (Sufficiency for a tangent model)**.** A channel $\mathcal{C}$ is sufficient for a tangent subspace $V_\rho \subseteq T_\rho \mathcal{D}_+$ with respect to g^f if equality holds in equation (43) for every $X \in V_\rho$.

This metric definition is weaker than full statistical sufficiency but is useful for local reconstruction. It says that the selected tangent information survives the channel. If equality fails, a geometric variable inferred from that tangent may cease to be identifiable at the coarser scale.

6.3 Why locality is additional structure

The same global algebra $M_{2^N}(\mathbb{C})$ admits many inequivalent tensor factorizations. A density operator and a monotone metric do not select one factorization. Even after a factorization is selected, they do not determine which factors are adjacent. A net or comparable incidence structure must therefore be part of the microscopic input.

This observation prevents a category error. Contractivity is an order relation on statistical distinguishability. Locality is a statement about where observables and variations are supported. One property does not imply the other.

7 The reconstruction interface

We now define the object that later work may attempt to construct in particular microscopic models. The definition is finite at the input and smooth at the output, so it already contains a continuum-limit hypothesis. That hypothesis is visible rather than being treated as a consequence of the information metric.

7.1 Input record

Definition 7.1 (Microscopic information datum)**.** At scale μ, a microscopic information datum is a tuple

$$\mathfrak{I}_\mu = (\mathsf{P}_\mu, \mathcal{A}_\mu, \Theta_\mu, \rho_\mu, f_\mu, \mathcal{C}_\mu) \tag{59}$$

with the following components.

(i) P_μ is a region poset.

(ii) $\mathcal{A}_\mu$ is a local net on P_μ.

(iii) Θ_μ is a smooth parameter manifold or a specified constant-rank stratum.

(iv) $\rho_{\mu,U} : \Theta_\mu \to \mathsf{S}(\mathcal{A}_\mu(U))$ is a compatible state family.

(v) f_μ specifies one Petz monotone metric, including normalization.

(vi) $\mathcal{C}_\mu$ is a compatible family of restriction and coarse-graining channels.

The metric choice f_μ may be fixed across scales or may itself be part of a scale-dependent modeling convention. Changing f_μ changes the input datum. It is not a coordinate change on a single metric.

7.2 Output record

Definition 7.2 (Geometric output datum)**.** A geometric output at scale μ is a tuple

$$\mathfrak{O}_\mu = (M_\mu, h_\mu, \tau_\mu, g_\mu, \Phi_\mu, \boldsymbol{c}_\mu) \tag{60}$$

where:

(i) M_μ is a smooth d-dimensional manifold.

(ii) h_μ is a positive Riemannian metric inferred from selected probe directions.

(iii) τ_μ is a nowhere-vanishing clock covector field or equivalent causal datum.

(iv) g_μ is a Lorentzian metric compatible with the chosen causal datum.

(v) Φ_μ denotes other surviving fields.

(vi) $\boldsymbol{c}_\mu$ denotes scale-dependent effective parameters.

The intermediate Riemannian metric h_μ is included to expose what a positive information metric can produce directly. The Lorentzian metric g_μ requires the additional causal datum τ_μ or some alternative signature-changing rule. One may replace the explicit clock construction below by another causal reconstruction, but that replacement remains an independent part of the interface.

7.3 Localization and probes

Definition 7.3 (Localization map)**.** A localization map is an order-preserving assignment

$$\ell_\mu : \mathsf{P}_\mu \longrightarrow \mathrm{Open}(M_\mu) \tag{61}$$

such that $U \subseteq V$ implies $\ell_\mu(U) \subseteq \ell_\mu(V)$.

Order preservation is weaker than an isomorphism of region lattices. Distinct microscopic regions may map to the same open set after coarse-graining. Some open sets may have no microscopic representative. Any stronger property must be stated separately.

Definition 7.4 (Probe map)**.** For each $x \in M_\mu$, a probe map is a linear map

$$J_{\mu,x} : T_x M_\mu \longrightarrow \mathcal{T}_{\mu,x}, \tag{62}$$

where $\mathcal{T}_{\mu,x}$ is a chosen quotient or subspace of localized microscopic tangent variations near x.

The quotient may remove microscopic parameter redundancies that act trivially on all selected local states. The information metric induces a positive bilinear form $\mathcal{G}_{\mu,x}$ on $\mathcal{T}_{\mu,x}$ whenever the support and rank assumptions are satisfied. The probe pullback is

$$h_{\mu,x}(v, w) = \mathcal{G}_{\mu,x}(J_{\mu,x} v, J_{\mu,x} w). \tag{63}$$

Definition 7.5 (Variation map)**.** A variation map is a linear map

$$V_{\mu,\lambda} : T_\lambda \Theta_\mu \longrightarrow \Gamma_{\mathrm{loc}}\big(S^2 T^* M_\mu \oplus E_\Phi\big) \tag{64}$$

that assigns localized variations of the reconstructed metric and other fields to microscopic state variations.

The probe map and variation map answer different questions. $J_{\mu,x}$ identifies candidate spacetime directions through localized distinguishing experiments. $V_{\mu,\lambda}$ identifies which geometric field variations are accessible when the microscopic state is changed. Neither map is determined by g^f alone.

7.4 Interface axioms

Definition 7.6 (Reconstruction interface)**.** An information-geometric reconstruction interface at scale μ is

$$\mathfrak{R}_\mu = (\mathfrak{I}_\mu, \mathfrak{O}_\mu, \ell_\mu, J_\mu, V_\mu, \mathcal{L}_\mu) \tag{65}$$

subject to explicitly selected conditions from the list below. The map $\mathcal{L}_\mu$ is the causal or Lorentzianization rule that relates (h_μ, τ_μ) to g_μ.

The principal conditions are as follows.

R1. Regularity. The selected state family stays in a smooth faithful or constant-rank stratum, and all output fields and interface maps have a stated differentiability class.

R2. Metric specification. The operator-monotone function f_μ, normalization, and any boundary prescription are fixed.

R3. Localization. The map ℓ_μ preserves inclusions and specifies how microscopic support is compared with open subsets of M_μ.

R4. Input locality. For each output open set O, there is a prescribed microscopic neighborhood $N_\mu(O)$ such that inputs agreeing on $N_\mu(O)$ produce the same output restricted to O.

R5. Probe identifiability. Each $J_{\mu,x}$ is injective after the declared microscopic redundancy quotient.

R6. Field identifiability. The kernel of $V_{\mu,\lambda}$ is exactly the declared class of state variations that are invisible to the retained infrared fields.

R7. Causal completion. A causal datum independent of the positive pullback is supplied and $\mathcal{L}_\mu$ produces a nondegenerate Lorentzian metric.

R8. Scale compatibility. For $\mu \preceq \nu$, the microscopic coarse-graining maps and geometric comparison maps form a commuting diagram up to a stated tolerance or equivalence.

R9. Redundancy covariance. Equivalent microscopic presentations give output data related by declared field redefinitions, including diffeomorphisms where appropriate.

R10. Controlled loss. If coarse-graining is not sufficient on the retained tangent model, the resulting loss of identifiability or resolution is quantified rather than silently ignored.

These are interface conditions, not universal properties of quantum state space. A concrete reconstruction may satisfy only a subset. Any theorem must name the subset it uses.

Definition 7.7 (Exact input locality)**.** Let $O \subseteq M_\mu$ be open. Suppose two microscopic data records $\mathfrak{I}_\mu$ and $\widetilde{\mathfrak{I}}_\mu$ agree after restriction to $N_\mu(O)$. The interface is exactly input-local if

$$\mathfrak{R}_\mu(\mathfrak{I}_\mu)|_O = \mathfrak{R}_\mu(\widetilde{\mathfrak{I}}_\mu)|_O. \tag{66}$$

In an approximate reconstruction, equation (66) may be replaced by a norm estimate

$$\left\| \mathfrak{R}_\mu(\mathfrak{I}_\mu)|_O - \mathfrak{R}_\mu(\widetilde{\mathfrak{I}}_\mu)|_O \right\| \le \varepsilon_\mu(O). \tag{67}$$

The norm, comparison gauge, and scale dependence of ε_μ must then be part of the definition.

8 Consequences of the explicit interface hypotheses

This section proves only statements that follow from the definitions above. No gravitational field equation is used.

8.1 Positive pullbacks and the signature obstruction

Proposition 8.1 (Positive pullback). *Let $(\mathcal{T}, \mathcal{G})$ be a positive definite real inner-product space and let $J : T_xM \to \mathcal{T}$ be linear. Then*

$$h_x(v, w) = \mathcal{G}(Jv, Jw) \tag{68}$$

is positive semidefinite. It is positive definite exactly when J is injective. In particular, if $\dim M > 1$, h_x cannot be a Lorentzian metric.

Proof. For every v, $h_x(v, v) = \mathcal{G}(Jv, Jv) \ge 0$. Equality holds exactly when $Jv = 0$ because $\mathcal{G}$ is positive definite. Thus h_x is nondegenerate and positive exactly when $\ker J = \{0\}$. A Lorentzian metric has one negative direction at every point, so it cannot equal a positive semidefinite pullback. □

Corollary 8.2 (No signature from monotonicity alone). *No choice of Petz monotone metric and linear probe pullback can by itself determine Lorentzian signature. An additional indefinite structure or signature-changing rule is necessary.*

This corollary is independent of which Petz metric is selected. Changing from BKM to SLD changes coefficients, but it does not change positivity.

8.2 A conditional Lorentzianization

One transparent causal completion uses a clock one-form. Let h be a Riemannian metric and let τ be a one-form normalized by

$$h^{-1}(\tau, \tau) = 1. \tag{69}$$

Write $T = \tau^{\sharp_h}$, so $h(T, T) = 1$ and $\tau(T) = 1$.

Theorem 8.3 (Clock Lorentzianization). *Let (M, h) be a smooth d-dimensional Riemannian manifold with $d \ge 2$. Let τ be a smooth one-form satisfying equation* (69)*. Then*

$$g = h - 2\tau \otimes \tau \tag{70}$$

is a smooth Lorentzian metric of signature $(-, +, \dots, +)$. The vector field $T = \tau^{\sharp_h}$ supplies a time orientation.

Proof. At any point, decompose $v = aT + s$ with $h(T, s) = 0$. Then $\tau(v) = a$ and

$$g(v, v) = h(aT + s, aT + s) - 2a^2 = -a^2 + h(s, s). \tag{71}$$

Thus the line spanned by T is negative, while its h-orthogonal complement is positive. The decomposition is direct, so g is nondegenerate with the claimed signature. Smoothness follows from smoothness of h and τ. The globally defined timelike field T determines a time orientation. □

Remark 8.4. The theorem is a sufficient construction, not a derivation of the clock field. Different normalized one-forms on the same (M, h) generally produce different light cones. The microscopic theory or an additional reconstruction rule must select among them.

Proposition 8.5 (Causal nonuniqueness). *Let (V, h) be a Euclidean vector space of dimension at least two. For two unit covectors τ and σ with $\sigma \neq \pm\tau$, the Lorentzian forms*

$$g_\tau = h - 2\tau \otimes \tau, \qquad g_\sigma = h - 2\sigma \otimes \sigma \tag{72}$$

are unequal and have different timelike axes. Therefore h alone does not select a causal structure.

Proof. The unique negative eigenspace of the h-self-adjoint map corresponding to g_τ is spanned by $\tau^{\sharp_h}$. The corresponding eigenspace for g_σ is spanned by $\sigma^{\sharp_h}$. If the forms were equal, these eigenspaces would agree, forcing $\sigma = \pm\tau$ after unit normalization. □

8.3 Identifiability

Proposition 8.6 (Probe identifiability criterion). *Assume $\mathcal{G}_{\mu,x}$ is positive definite on the microscopic tangent quotient $\mathcal{T}_{\mu,x}$. Then the probe pullback $h_{\mu,x}$ in equation (63) is a Riemannian metric exactly when*

$$\ker J_{\mu,x} = \{0\}. \tag{73}$$

If $J_{\mu,x}$ has a nonzero kernel, the information data cannot distinguish all candidate spacetime directions at x.

Proof. This is the nondegeneracy part of theorem 8.1 applied pointwise. The operational interpretation follows because a vector in the kernel maps to a zero localized state variation and hence has zero length in every bilinear pairing with the selected probes. □

The proposition gives a rank test. It does not prove that the image of J has a geometrically local interpretation. That interpretation is supplied by ℓ_μ and the input-locality condition.

Proposition 8.7 (Coarse-graining can destroy identifiability). *Let $J_\mu : T_xM \to \mathcal{T}_{\mu,x}$ be injective and let $C : \mathcal{T}_{\mu,x} \to \mathcal{T}_{\nu,x}$ be the tangent map of a coarse-graining channel. The coarser probe $J_\nu = CJ_\mu$ is injective exactly when*

$$\operatorname{im} J_\mu \cap \ker C = \{0\}. \tag{74}$$

Proof. A vector v lies in $\ker J_\nu$ exactly when $J_\mu v \in \ker C$. Since J_μ is injective, nonzero v correspond exactly to nonzero elements of $\operatorname{im} J_\mu$. This proves the equivalence. □

Metric contraction is not enough to prevent the intersection in equation (74). A channel may contract a selected direction all the way to zero. Thus scale compatibility needs a sufficient-statistics or rank condition in addition to monotonicity.

8.4 Locality under composition

Lemma 8.8 (Composition of exact local reconstructions)**.** *Suppose a reconstruction $\mathfrak{R}_1$ maps microscopic data to an intermediate field and depends on $N_1(O)$ for its output on O. Suppose $\mathfrak{R}_2$ maps the intermediate field to a final field and depends on $N_2(O)$ for its output on O. Then $\mathfrak{R}_2 \circ \mathfrak{R}_1$ depends only on*

$$N_{12}(O) = \bigcup_{O' \subseteq N_2(O)} N_1(O'). \tag{75}$$

Proof. If two microscopic inputs agree on $N_{12}(O)$, the intermediate outputs agree on every O' needed to determine $\mathfrak{R}_2$ on O. Exact locality of $\mathfrak{R}_2$ then makes the final outputs agree on O. □

The neighborhood can grow under repeated coarse-graining. Locality at each step does not imply a scale-independent localization radius. Quantifying that growth is part of a continuum-limit analysis.

8.5 A conditional scale comparison

Let $P_{\nu\leftarrow\mu} : M_\mu \to M_\nu$ be a smooth comparison map between output manifolds. Let $C_{\nu\leftarrow\mu}$ be the microscopic tangent channel. A strict probe compatibility condition is

$$C_{\nu\leftarrow\mu} \circ J_{\mu,x} = J_{\nu,P(x)} \circ (\mathrm{d}P)_x. \tag{76}$$

Proposition 8.9 (Metric comparison under a commuting probe square)**.** *Assume equation* (76) *and use the same Petz metric convention at both scales. Then for every $v \in T_x M_\mu$,*

$$h_{\nu,P(x)}\big((\mathrm{d}P)v, (\mathrm{d}P)v\big) \le h_{\mu,x}(v, v). \tag{77}$$

Proof. Use equation (76) to rewrite the left side as the coarse metric applied to $CJ_{\mu,x}v$. Apply theorem 4.7 to obtain the right side. □

The result compares the positive metrics h_μ. It does not compare Lorentzian metrics unless the causal data and Lorentzianization maps also commute with scale change.

9 Finite examples and counterexamples

The examples in this section are small enough to compute exactly. Their role is to test definitions and expose missing hypotheses. They are not models of continuum gravity.

9.1 A binary classical family

Let

$$p(\theta) = (\theta, 1-\theta), \qquad 0 < \theta < 1. \tag{78}$$

The tangent for unit parameter velocity is $u = (1, -1)$. The Fisher coefficient is

$$g^{\mathrm{F}}_{\theta}(u, u) = \frac{1}{\theta} + \frac{1}{1-\theta} = \frac{1}{\theta(1-\theta)}. \tag{79}$$

It diverges at the boundary because changing a probability away from zero can become perfectly distinguishable at first order.

Consider the binary symmetric channel with crossover probability $0 \le \epsilon \le 1/2$:

$$K_{\epsilon} = \begin{pmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{pmatrix}. \tag{80}$$

The output parameter is

$$q(\theta) = \epsilon + (1 - 2\epsilon)\theta \tag{81}$$

and its derivative is $1 - 2\epsilon$. The output Fisher coefficient is

$$I_{\mathrm{out}}(\theta) = \frac{(1-2\epsilon)^2}{q(\theta)(1-q(\theta))}. \tag{82}$$

At $\epsilon = 1/2$, the channel maps every input to $(1/2, 1/2)$ and identifiability is completely lost. This realizes theorem 8.7 in one dimension.

9.2 Commuting density operators

Embed the binary family as diagonal qubit states

$$\rho(\theta) = \begin{pmatrix} \theta & 0 \\ 0 & 1-\theta \end{pmatrix}. \tag{83}$$

For the diagonal tangent $X = \mathrm{diag}(1, -1)$,

$$g^{\mathrm{BKM}}_{\rho}(X, X) = g^{\mathrm{SLD}}_{\rho}(X, X) = \frac{1}{\theta(1-\theta)}, \tag{84}$$

while

$$g^{\mathrm{Bu}}_{\rho}(X, X) = \frac{1}{4\theta(1-\theta)}. \tag{85}$$

The factor is purely the Bures normalization chosen in equation (35). The equality between BKM and SLD here is caused by commutativity and does not persist for the off-diagonal tangent in theorem 4.8.

9.3 A two-site locality counterexample

Let the global algebra be $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$. One region poset declares the two tensor factors to be adjacent sites. Apply a global unitary U that does not factor into site unitaries and use

$$\widetilde{\mathcal{A}}_1 = U(M_2 \otimes I)U^*, \qquad \widetilde{\mathcal{A}}_2 = U(I \otimes M_2)U^*. \tag{86}$$

The same global state space and every unitarily invariant Petz metric admit both factorizations. Nothing in the metric selects which pair should be called spatially local. The region net is therefore indispensable input.

9.4 A positive metric with two causal completions

Let the inferred information metric on $\mathbb{R}^2$ be

$$h = \mathrm{d}x^2 + \mathrm{d}y^2. \tag{87}$$

Choosing $\tau = \mathrm{d}x$ gives

$$g_x = -\mathrm{d}x^2 + \mathrm{d}y^2. \tag{88}$$

Choosing $\sigma = \mathrm{d}y$ gives

$$g_y = \mathrm{d}x^2 - \mathrm{d}y^2. \tag{89}$$

Both are Lorentzian, but their timelike axes differ. The positive metric does not favor one choice.

9.5 A redundant parameterization

Let $\rho(\lambda^1, \lambda^2) = \rho(\lambda^1 + \lambda^2)$ for a one-parameter faithful family. The parameter tangent $(1, -1)$ maps to zero. Every state-space information metric pulled back to the two-dimensional parameter domain is degenerate along this direction. Calling the parameter domain a reconstructed surface would therefore fail probe identifiability. Taking the quotient by the redundant direction restores a one-dimensional Riemannian metric, but it still supplies no Lorentzian spacetime of dimension two or higher.

10 Relation to established reconstruction settings

The interface is broad enough to compare several established constructions, but it does not identify them. Each setting supplies different pieces of the record in equation (65).

10.1 Classical statistical models

For a regular classical model, Chentsov's theorem characterizes the Fisher metric, up to scale, by invariance under sufficient statistics [13]. The result concerns statistical models and stochastic maps. It does not construct a spacetime manifold or a causal metric. In the present language it fixes much of the metric specification condition while leaving localization, probes, and causal completion open.

10.2 Quantum monotone geometry

Petz's classification replaces classical uniqueness by an operator-monotone family. A reconstruction must choose a member of this family according to its operational question. The BKM metric is natural for relative entropy and thermodynamic response. The SLD metric is natural for single-parameter local estimation. The Bures metric is natural when the Bures distance or purification geometry is primary. These motivations can overlap, but the tensors must not be substituted without an explicit dictionary.

Lesniewski and Ruskai relate monotone Riemannian metrics to noncommutative relative entropies and contraction coefficients [10]. Their framework reinforces the need to fix the contrast function and its associated operator mean. It does not remove the choice.

10.3 Finite entanglement-graph reconstruction

Cao, Carroll, and Michalakis begin with a chosen tensor factorization of Hilbert space, use mutual information to weight a graph of factors, and apply multidimensional scaling to seek a best-fit spatial geometry [11]. Their construction makes two pieces of the present interface especially visible. The tensor factorization and graph supply localization data that a global density operator does not select by itself. The recovered metric is spatial and positive, while time evolution and spacetime dynamics require further physical input. This is consistent with separating the positive probe pullback from causal completion.

10.4 Algebraic locality

The Haag-Kastler framework begins with a net of local algebras on an already given spacetime [14]. Our input net is more primitive because its indexing poset need not yet be identified with spacetime opens. The localization map ℓ_μ is precisely the extra comparison needed to pass between the two index structures. Proving that it is sufficiently faithful, covers a manifold, and respects causal disjointness would be a substantial model-dependent theorem.

10.5 Holographic relative entropy

In holographic code subspaces, boundary relative entropy can equal bulk relative entropy for corresponding regions [20]. The second variation then relates boundary quantum Fisher information to bulk canonical energy in the setting analyzed by Lashkari and Van Raamsdonk [21]. These are strong reconstruction dictionaries in a particular domain. They provide more than monotonicity because they identify regions, state variations, and a bulk gravitational quadratic form.

The present paper does not extend those equalities to arbitrary quantum systems. Instead, it identifies where comparable data would enter a general interface. The boundary-region assignment contributes to ℓ_μ. The code-subspace map contributes to V_μ. The canonical-energy equality gives a model-specific relation between a microscopic information tensor and a geometric response form. None of these relations follows from the Petz classification alone.

10.6 Entanglement and connectivity

Ryu and Takayanagi related boundary entanglement entropy to bulk extremal area in static holographic settings [15]. Van Raamsdonk emphasized the relation between entanglement patterns and connected bulk geometry [16]. Swingle connected entanglement renormalization with an emergent holographic direction [17]. Quantum error-correcting models make aspects of bulk reconstruction explicit [18, 19].

These works motivate candidate localization and scale maps. They do not imply that every monotone metric on every local state family yields a smooth Lorentzian manifold. In particular, the causal completion problem remains logically distinct from the positive information metric.

11 What can and cannot be inferred

The formal distinction between data and consequences makes several common inferences easy to audit.

11.1 Consequences available from the microscopic metric

Once a faithful family and a Petz metric are fixed, the following statements are available. The tangent bilinear form is positive. It contracts under completely positive trace-preserving maps. It reduces to classical Fisher information on commuting directions, subject to normalization. Its pullback along a probe map is positive semidefinite. Its degeneracy detects directions invisible to the selected state model.

For the BKM choice, the tensor is the correctly signed mixed Hessian of Umegaki relative entropy. For the SLD choice, the tensor has a direct local-estimation interpretation. For the Bures choice, it is the quadratic coefficient of the Bures distance and is one quarter of SLD on a smooth constant-rank stratum under our conventions.

11.2 Data not supplied by monotone information geometry

A monotone metric does not select a tensor factorization or a region poset. It does not construct a manifold from the poset. It does not establish that the output dimension is stable under scale change. It does not select a Lorentzian time direction. It does not produce diffeomorphism redundancy. It does not identify a stress tensor or an effective action. It does not imply an Einstein equation.

Each of those steps requires additional structure. Later papers may study consequences of such structure, but those consequences remain conditional on the interface hypotheses.

11.3 Finite dimension and operator-algebraic extensions

Most proofs in this paper are stated in finite dimension to avoid domain questions. For von Neumann algebras, relative entropy can be defined using modular theory, as in Araki's formulation [4]. Unbounded modular generators, type III algebras, and the absence of density matrices require a different analytic treatment. The finite-dimensional formulas should not be transferred term by term without checking those domains.

The local-net definition is likewise only a finite model. Continuum algebraic quantum field theory adds isotony, locality, covariance, spectral conditions, and other structure. A continuum reconstruction must explain how the microscopic net converges to, or represents, the relevant operator-algebraic object.

11.4 Rank and regularity

Faithfulness is not merely technical convenience for BKM geometry. It fixes the domain of the logarithm and gives a smooth manifold on which the tensor varies regularly. If physical states cross rank strata, the reconstruction must specify which boundary extension is intended. The rank-changing example in theorem 5.1 shows that different natural extensions can disagree.

11.5 Identifiability is scale dependent

An injective probe at one scale can become noninjective after a lossy channel. Conversely, a coarse model may discard nuisance directions while retaining the tangent subspace relevant to a chosen field. The correct condition is therefore not preservation of all microscopic information. It is preservation of the declared geometric tangent model with quantified error.

11.6 Lorentzian signature is not a negative information direction

The negative direction in a Lorentzian metric should not be described as negative statistical distinguishability. Statistical distinguishability remains nonnegative. In theorem 8.3, the sign change is applied by an independent causal rule after a positive Riemannian form has been reconstructed. This separation avoids interpreting a timelike interval as a negative variance or negative Fisher information.

12 A minimal reconstruction problem

The interface suggests a concrete problem that is narrower than a universal emergence claim. Fix a sequence of finite local nets $\mathcal{A}_\mu$, compatible faithful state families ρ_μ, and one monotone metric g^f. Seek manifolds M_μ, localization maps ℓ_μ, and injective probes J_μ such that the pullback metrics h_μ converge in a stated topology. Independently seek causal covectors τ_μ or a comparable causal reconstruction whose Lorentzian metrics also converge. Finally require the scale comparison square equation (76) to commute approximately with an error that vanishes in the limit.

This problem contains several separately testable questions. Does the incidence structure admit a stable dimension? Do the localized tangent quotients have enough rank for injective probes? Does coarse-graining preserve the relevant tangent subspaces? Is the limit metric smooth and nondegenerate? Can the causal datum be inferred from a microscopic order, modular flow, or another independent observable? Are different reconstruction choices related by a declared redundancy?

Conjecture 12.1 (Model-dependent convergence target)**.** Let $\{\mathfrak{I}_\mu\}$ be a scale-indexed family of microscopic information data with uniformly controlled locality neighborhoods. Suppose there are compatible injective probes whose pullback metrics have uniform local regularity bounds and converge to a Riemannian metric h on a smooth manifold M. Suppose further that compatible normalized clock fields converge to τ. Then the Lorentzianized metrics $g_\mu = h_\mu - 2\tau_\mu \otimes \tau_\mu$ converge to $g = h - 2\tau \otimes \tau$ in the corresponding local topology.

The final implication in the conjecture is elementary once a common manifold and topology are fixed. The difficult content is the existence of the manifold, localization maps, probes, regularity bounds, and compatible clocks. Those hypotheses are not consequences of state-space information geometry.

13 Discussion

The central choice in microscopic information geometry is not a label but an operator mean. Umegaki relative entropy selects the logarithmic mean and hence the BKM tensor. The symmetric logarithmic derivative selects the arithmetic mean. Bures distance supplies a metric proportional to the SLD tensor only under a stated normalization and on a regular rank stratum. General Petz metrics interpolate through other symmetric operator-monotone functions.

This plurality is useful. Different metrics encode different operational tasks. It becomes harmful only when equations proved for one tensor are silently transferred to another. A reconstruction paper should therefore state at least four items whenever it writes an information metric: the state domain, the defining contrast or operator-monotone function, the normalization, and the rank convention.

The reconstruction interface adds an equally important second list. It asks for a region structure, a localization map, a probe map, a field-variation map, a causal completion, and scale comparison data. These maps make it possible to formulate locality and identifiability as mathematical conditions. Without them, the statement that a spacetime metric is reconstructed from distinguishability is underdetermined.

The positive-signature obstruction is elementary but consequential. No refinement of a positive Petz pullback can turn it into a Lorentzian form while preserving the same linear pullback construction. One may obtain a Lorentzian metric by adding causal order, modular time, a clock field, an analytic continuation rule, or another indefinite structure. The choice may be physically well motivated in a concrete theory. It is still extra data.

The clock construction was chosen because its assumptions and proof are transparent. It should not be mistaken for a universal microscopic mechanism. A future model could instead reconstruct conformal cones first and determine a volume element from information data. Another could use causal order plus a measure. The interface accommodates such variants by replacing $\mathcal{L}_\mu$ while retaining the demand that its inputs be named.

Coarse-graining supplies inequalities rather than geometry by itself. Metric contraction says that a channel cannot improve local discrimination under the selected monotone metric. For reconstruction, the sharper issue is whether the channel is sufficient on the image of the probe map. That is exactly the intersection condition in equation (74). It provides a finite-dimensional diagnostic for when a candidate geometric coordinate disappears at a coarser scale.

No dynamical equation has been derived here. To vary an infrared metric in an effective action, one would need to know that the variation map in equation (64) reaches the admissible local metric variations modulo the relevant redundancies. That is a further hypothesis on the reconstruction. Information-metric positivity alone does not supply it.

14 Conclusion

A smooth faithful quantum state family supports a classified family of monotone Riemannian metrics. The mixed Hessian of Umegaki relative entropy is the BKM metric when the mixed derivative carries a minus sign. The SLD Fisher tensor uses a different operator mean. The

Bures tensor is one quarter of SLD under the distance convention adopted here on smooth constant-rank strata. Rank changes require a separate prescription.

Geometric reconstruction needs data that these metrics do not contain. A local net supplies subsystem incidence. Localization and probe maps compare microscopic variations with candidate manifold directions. Injectivity makes the positive pullback nondegenerate. An independent causal datum is required for Lorentzian signature. Scale compatibility controls which distinguishing directions survive coarse-graining.

These ingredients define a precise input to later infrared arguments. Their existence remains a model-dependent mathematical and physical problem.

A Derivative identities for the BKM metric

This appendix records finite-dimensional identities used in the main text. Let $A > 0$ and let X, Y be Hermitian. The resolvent formula for the logarithm is

$$\log A = \int_0^\infty \left(\frac{1}{1+s} I - (A + sI)^{-1} \right) \mathrm{d}s. \tag{90}$$

Differentiating under the integral gives equation (21).

The derivative is self-adjoint in the Hilbert-Schmidt inner product:

$$\mathrm{Tr}\left[X (\mathrm{d}\log)_A(Y)\right] = \mathrm{Tr}\left[Y (\mathrm{d}\log)_A(X)\right]. \tag{91}$$

This follows either from cyclicity under the resolvent integral or from the symmetric divided difference in equation (24).

Positivity is explicit:

$$\mathrm{Tr}\left[X (\mathrm{d}\log)_A(X)\right] = \sum_{i,j} c_{\mathrm{BKM}}(a_i, a_j) |X_{ij}|^2 > 0 \tag{92}$$

for nonzero Hermitian X. The strict inequality uses $a_i > 0$.

The Kubo-Mori operator in equation (22) has eigenbasis coefficients

$$[\Omega_A(Y)]_{ij} = m_{\log}(a_i, a_j) Y_{ij}, \tag{93}$$

where the logarithmic mean is

$$m_{\log}(x, y) = \begin{cases} \dfrac{x - y}{\log x - \log y}, & x \neq y, \\ x, & x = y. \end{cases} \tag{94}$$

Thus $c_{\mathrm{BKM}} = m_{\log}^{-1}$ entrywise.

For a curve $\rho(t) = \rho + tX + O(t^2)$ with $\mathrm{Tr}\, X = 0$, the one-sided expansion in the second argument is

$$D(\rho \| \rho(t)) = \frac{t^2}{2} g_\rho^{\mathrm{BKM}}(X, X) + O(t^3). \tag{95}$$

The first-order term is

$$-t\,\mathrm{Tr}\,[\rho(\mathrm{d}\log)_\rho(X)] = -t\,\mathrm{Tr}\,X = 0. \tag{96}$$

For two varying arguments, let

$$\rho(s) = \rho + sX + O(s^2), \qquad \sigma(t) = \rho + tY + O(t^2). \tag{97}$$

The st coefficient in $D(\rho(s)\|\sigma(t))$ is

$$-g_\rho^{\mathrm{BKM}}(X, Y). \tag{98}$$

Indeed, after differentiating the second argument, the first state occurs linearly in

$$-\,\mathrm{Tr}\left[\rho(s)(\mathrm{d}\log)_{\sigma(t)}(Y)\right]. \tag{99}$$

The s derivative therefore inserts X directly before one sets $\sigma(t) = \rho$. This is the concrete source of the mixed-Hessian sign.

B Spectral comparison of BKM and SLD

For positive x and y, the logarithmic mean satisfies

$$\sqrt{xy} \le m_{\log}(x, y) \le \frac{x+y}{2}. \tag{100}$$

Taking reciprocals gives

$$\frac{2}{x+y} \le c_{\mathrm{BKM}}(x, y) \le \frac{1}{\sqrt{xy}}. \tag{101}$$

The left coefficient is the SLD kernel. Consequently,

$$g_\rho^{\mathrm{SLD}}(X, X) \le g_\rho^{\mathrm{BKM}}(X, X) \tag{102}$$

for faithful finite-dimensional states under the normalizations used here. Equality holds for matrix entries connecting equal eigenvalues and, in particular, for all commuting tangent directions.

For the off-diagonal qubit tangent in theorem 4.8, set $p = 1/4$. Then

$$g_\rho^{\mathrm{BKM}}(X, X) = 4|x|^2 \log 3, \qquad g_\rho^{\mathrm{SLD}}(X, X) = 4|x|^2. \tag{103}$$

The ratio is $\log 3$. At $p = 1/2$, the eigenvalues coincide and the ratio tends to one.

This coefficient comparison should not be read as a universal ordering of every normalized convention found in the literature. Multiplying a metric by a conventional constant changes the inequality. The operator kernels in equation (41) and equation (42) fix the convention used here.

C A finite reconstruction record

For computational examples it is useful to replace smooth bundles by finite matrices. Let P be a finite region poset and let Q_U be a finite-dimensional tangent vector space for each $U \in P$. Let G_U be a positive matrix representing a chosen information metric. Let J_x be a matrix from a candidate geometric tangent space $\mathbb{R}^d$ to a localized direct sum of the Q_U. Then

$$h_x = J_x^{\mathsf{T}} G_x J_x. \tag{104}$$

The matrix h_x is positive semidefinite. It is positive definite exactly when J_x has column rank d. This gives a direct numerical identifiability test. If a coarse-graining tangent matrix is C_x, then

$$h'_x = J_x^{\mathsf{T}} C_x^{\mathsf{T}} G'_x C_x J_x. \tag{105}$$

Metric monotonicity implies $h'_x \preceq h_x$ as quadratic forms. Full rank of h_x does not imply full rank of h'_x.

In two dimensions, suppose

$$h = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{106}$$

with $a > 0$ and $ac - b^2 > 0$. Choose a covector τ satisfying

$$\tau^{\mathsf{T}} h^{-1} \tau = 1. \tag{107}$$

Then

$$g = h - 2\tau\tau^{\mathsf{T}} \tag{108}$$

has determinant

$$\det g = \det h \left(1 - 2\tau^{\mathsf{T}} h^{-1} \tau\right) = -\det h < 0, \tag{109}$$

where the matrix determinant lemma was used. A real symmetric two by two matrix with negative determinant has one positive and one negative eigenvalue.

This finite record is suitable for testing arithmetic and rank conditions. It does not establish convergence to a smooth manifold. It also does not determine the clock covector.

D Checklist for a model-specific reconstruction claim

A model-specific claim can be made auditable by answering the following questions in mathematical terms.

1. What are the local algebras or subsystems, and what is their incidence relation?
2. What state family is varied, and on which faithful or constant-rank domain?
3. Which divergence or operator-monotone function fixes the metric?
4. What normalization is used for Bures or SLD quantities?
5. How are rank-changing points treated?

6. Which coarse-graining channels act on states and tangent operators?
7. What is the candidate output manifold and how is its dimension obtained?
8. Which microscopic regions correspond to which output open sets?
9. What linear probe map connects tangent directions?
10. Which microscopic redundancies are quotiented before testing injectivity?
11. How are geometric field variations induced from state variations?
12. What additional datum determines causal cones and time orientation?
13. Which diagrams commute across scale, and in what topology or error norm?
14. What evidence establishes smoothness and nondegeneracy?
15. Which conclusions are exact and which depend on a continuum approximation?

An answer consisting only of a density matrix and a Fisher metric does not determine the remaining items. Conversely, a concrete model that supplies these maps permits precise tests even before any dynamical field equation is considered.

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